

2nd homework set, Due !!!April 7, 10:15!!!

1. (2p.) Find out whether there exists a binary prefix code with code lengths

- (a) 2,3,3,3,4,4,4,5,5,5,5,5,6,7,7
- (b) 2,3,3,4,4,4,4,4,4,5,5,5,5,6

If yes, then define such a coding!

2. (1,5p.) For a closed convex set Π of distributions on A , show that P^* maximizes $H(\mathbb{P})$ subject to $\mathbb{P} \in \Pi$ if and only if P^* is the I-projection of the uniform distribution on A onto Π , and that then

$$D(\mathbb{P}||\mathbb{P}^*) \leq H(\mathbb{P}^*) - H(\mathbb{P}), \text{ for all } \mathbb{P} \in \Pi. \quad (1)$$

3. (2p.+ 2p. + 3.5p.) (Hypothesis testing with both errors exponentially decreasing)

- (a) We observe independent drawings from an unknown distribution \mathbb{Q} on the finite set A . Let γ be a positive number and let \mathbb{P}_0 and \mathbb{P}_1 be strictly positive distributions on A with $D(\mathbb{P}_1||\mathbb{P}_0) > \gamma$. To test the (simple) null hypothesis $\mathbb{Q} = \mathbb{P}_0$ against the simple alternative hypothesis $\mathbb{Q} = \mathbb{P}_1$, let the acceptance region $A_n \subset \mathcal{X}^n$ be the union of all type classes $|\mathcal{T}_{\mathbb{P}}^n|$ with $D(\mathbb{P}||\mathbb{P}_0) \leq \gamma$. Show that then the probability of type 1 error decreases with exponent γ , i.e.,

$$\mathbb{P}_0^n(\mathcal{X}^n - A_n) = 2^{-n\gamma + o(n)} \quad \left(\text{or, i.e., } \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_0^n(\mathcal{X}^n - A_n) = -\gamma \right), \quad (2)$$

whereas the type 2 error probability ($\mathbb{P}_1^n(A_n)$) decreases with exponent $\delta = D(\mathbb{P}^*||\mathbb{P}_1)$ where \mathbb{P}^* is the I-projection of \mathbb{P}_1 onto the "divergence ball"

$$B(\mathbb{P}_0, \gamma) = \{\mathbb{P} : D(\mathbb{P}||\mathbb{P}_0) \leq \gamma\}. \quad (3)$$

Hint: Apply Sanov's theorem! Note that $B(\mathbb{P}_0, \gamma)$ is closed and its interior is $\{\mathbb{P} : D(\mathbb{P}||\mathbb{P}_0) < \gamma\}$.

- (b) Show that the above is the best possible, i.e., for any $\tilde{A}_n \subset \mathcal{X}^n$ satisfying (2), always

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_1^n(\tilde{A}_n) \geq -\delta. \quad (4)$$

Hint: Fix an $\varepsilon > 0$. (2) implies that $\exists N$ such that $\mathbb{P}_0^n(\mathcal{X}^n - \tilde{A}_n) \leq 2^{-n(\gamma-\varepsilon)}$ if $n > N$. Let Q be an arbitrary n -type in $B(\mathbb{P}_0, \gamma - 2\varepsilon)$. Show that \tilde{A}_n contains at least half of \mathcal{T}_Q^n if n is large enough!

- (c) With the notation used above, show that the I-Projection \mathbb{P}^* of \mathbb{P}_1 onto $B(\mathbb{P}_0, \gamma)$ equals the I-projection of both \mathbb{P}_0 and \mathbb{P}_1 onto the linear family

$$\mathcal{L} = \left\{ \mathbb{P} : \sum_{a \in A} \mathbb{P}(a) \log \frac{\mathbb{P}_0(a)}{\mathbb{P}_1(a)} = \delta - \gamma \right\} = \left\{ \mathbb{P} : D(\mathbb{P}||\mathbb{P}_1) - D(\mathbb{P}||\mathbb{P}_0) = \delta - \gamma \right\}, \quad (5)$$

and also equals the I-projection of \mathbb{P}_0 onto $B(\mathbb{P}_1, \delta)$. Give a geometric interpretation. Finally conclude that \mathbb{P}^* is of the form $\mathbb{P}^*(a) = c \cdot \mathbb{P}_0^\theta(a) \cdot \mathbb{P}_1^{1-\theta}(a)$ for some $0 < \theta < 1$.

Hint: We learned that $D(\mathbb{Q}||\mathbb{P})$ is strictly convex in \mathbb{Q} when \mathbb{P} is fixed and strictly positive. Using this fact first prove that \mathbb{P}^* is on the border of $B(\mathbb{P}_0, \gamma)$, i.e., $D(\mathbb{P}^*||\mathbb{P}_0) = \gamma$! After that, prove that $B(\mathbb{P}_0, \gamma) \cap \mathcal{L} = \{\mathbb{P}^*\}$! Then prove that the I-projection of \mathbb{P}_1 onto \mathcal{L} equals \mathbb{P}^* . Finally, prove the remaining statements!

As you can see you, can get one extra point for the second homework! Have a nice work!