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The dimension theory of some special families of self-similar fractals of overlapping construction satisfying the Weak Separation Property.

Thesis

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Budapest, 2015

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Introduction

In my thesis I will focus on iterated function systems construated with overlapping parts. These IFSs are from a special family, namely where the linear part in each map are the same. This linear part is a concrate contraction which can be interpreted as a reciprocal of a natural number, and the translations are chosen from a lattice in $\mathbb{Z}^d$.

This document first gives a brief overview in Section 1 of the corresponding results on geometric measure theory. The following sections will examine the literature about the topics, including the studies done by Richard Kenyon [3] on the projection of the Sierpiński carpet. A more recent study from 2000 by Lau, Ngai and Rau [6] gave a matrix expression which fully represents an IFS. Section 3 shows a method introduced by Víctor Ruiz [5] to investigate the relation between Hausdorff dimensions and absolute continuity.

In Section 5 we extend the results of Bárány, Rams [4, Theorem 1.2], concerning orthogonal projections of the Sierpiński carpets.

1 Preliminaries

Here we introduce some definitions, properties and well known results. In this section we follow the book in preparation by K. Simon and B. Solomyak [10].

Self-similar measures

For some integers $m \geq 2$ and $d \geq 1$, we call the collection $\mathcal{S} = \{S_1, \ldots, S_m\}$ of m contracting similarity transformations acting on $\mathbb{R}^d$ a *self-similar IFS* on $\mathbb{R}^d$ with *contraction ratios* $0 < r_i < 1, i = 1, \ldots, m$ if

$$\forall i \leq m,\ \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^d,\ ||S_i(\mathbf{x}) - S_i(\mathbf{y})|| = r_i||\mathbf{x} - \mathbf{y}||.$$

The IFS $\{f_1, \ldots, f_m\}$ on $\mathbb{R}^d$ is defined by

$$f_i(x) = r_i M_i x + b_i,\ i = 1, \ldots, m,$$

where $b_i \in \mathbb{R}^d$ and M_i is a $d \times d$ orthogonal matrix. Let Ω be the set $\{1, \ldots, m\}$, and let Ω^k denote the set of all words of length k in Ω, and let $\Omega^* = \bigcup_{k=1}^{\infty} \Omega^k$ denote the set of all finite words in Ω. For $\boldsymbol{i} \in \Omega^k, \boldsymbol{j} \in \Omega^n$, let $\boldsymbol{ij} \in \Omega^{k+n}$ denote the concatenation of $\boldsymbol{i}$ and $\boldsymbol{j}$.

In the above case we choose our favourite non-empty compact set H satisfying

$$S_i(H) \subset H\ \forall\ i = 1, \ldots, m.$$

We can always choose H as a large enough closed ball

$$B := \bar{B}(\mathbf{0}, R) \text{ for } R := \max_i \left\{ \frac{||S_i(\mathbf{0}||}{1 - r_i} \right\}. \tag{1}$$

Definition 1. *The set of points* $\bigcup_{i_1,\ldots,i_{n+1}} S_{i_1,\ldots,i_{n+1}}(H)$ *is a decreasing sequence of non-empty compact sets.*

Their intersection, the attractor Λ*, which can be interpreted as the set of all points that remain after infinetely many iterations:*

$$\Lambda := \bigcap_{n=1}^{m} \bigcup_{(i_1 \ldots i_n) \in \{1,\ldots,m\}^n} S_{i_1 \ldots i_n}(H), \tag{2}$$

where $S_{i_1 \ldots i_n}$ *is the level* n cylinders.

Suppose we are given a probability vector $\boldsymbol{p} = (p_1, \ldots, p_m)$ and a self-similar IFS $\mathcal{S} = \{S_1, \ldots, S_m\}$ on $\mathbb{R}^d$ with contracton ratios $0 < r_i < 1$.

Definition 2. *For an IFS $\mathcal{F} = \{f_i\}_{i=1}^m$ we denote the symbolic space by $\Sigma := \{1,\ldots,m\}^{\mathbb{N}}$ and the elements of Σ by $\boldsymbol{i} = (i_1, i_2, \ldots)$, $\boldsymbol{j} = (j_1, j_2, \ldots)$.*

The natural projection $\Pi\colon \Sigma \to \Lambda$ is defined by

$$\Pi(\boldsymbol{i}) := \lim_{n\to\infty} f_{i_1\ldots i_n}(\mathbf{0}) = \bigcap_{n=1}^{\infty} f_{i_1\ldots i_n}(B), \tag{3}$$

for the closed ball B defined in Equation (1). Let

$$\Lambda_{i_1\ldots i_n} := S_{i_1\ldots i_n}(\Lambda)$$

and call these sets the level-n cylinders of the attractor Λ.

Consider the *push-down measure* of the infinite product measure

$$\boldsymbol{p}^{\mathbb{N}} := (p_1, \ldots p_m)^{\mathbb{N}} : \ \nu := \Pi_* \boldsymbol{p}^{\mathbb{N}},$$

that is, $\nu(A) := \boldsymbol{p}^{\mathbb{N}}(\Pi^{-1}(A))$ for Borel $A \subset \mathbb{R}^d$. We say that ν is the *invariant measure* (*stationary measure, self-similar measure*) for the IFS $\mathcal{S}$ with probability vector $\boldsymbol{p}$. The support $\mathrm{spt}(\nu)$ of an invariant measure ν is Λ. Observe that the simplest example of a self-similar measure is the restriction of the Lebesgue measure to the interval $[0,1]$.

By considering the self-similar measure ν with probability vector $\boldsymbol{p}$ the following equation holds for all Borel set $A \subset \mathbb{R}^d$:

$$(A) = p_1\nu(S_1^{-1}(A)) + \cdots + p_m\nu(S_m^{-1}(A)). \tag{4}$$

Alternatively we can view ν as a fixed point of the operator:

$$\mathfrak{F}_{S,\boldsymbol{p}}\colon \nu \mapsto \sum_{k=1}^{m} p_k \cdot (\nu \circ S_k^{-1}),$$

which acts on an appropriately chosen space of Borel probability measures.

Definition 3. *For a self-similar IFS and for a probability vector $\boldsymbol{p}$, the only Borel probability measure satisfying Equation (4) is the self-similar measure $\nu = \Pi_*\boldsymbol{p}^{\mathbb{N}}$. In this way, Equation (4) can serve as an equivalent definition of the self-similar measures.*

Hausdorff and similarity dimension [14]

Definition 4 (Box dimension)**.** *Let $E \subset \mathbb{R}^d, E \neq \emptyset$, bounded. $N_\delta(E)$ be the smallest number of sets of diameter δ which can cover E. Then the* lower *and* upper box dimensions *of E:*

$$\underline{\dim}_B(E) := \liminf_{r\to 0} \frac{\log N_\delta(E)}{-\log\delta}, \tag{5}$$

$$\bar{\dim}_B(E) := \limsup_{r\to 0} \frac{\log N_\delta(E)}{-\log\delta}. \tag{6}$$

If the limit exists then we call it the box dimension *of* E.

Definition 5 (Hausdorff measure on $\mathbb{R}^d$)**.** *Let* $\Lambda \subset \mathbb{R}^d$ *and let* $t \geq 0$*. We define*

$$\mathcal{H}^t(\Lambda) = \lim_{\delta\to 0} \left\{\mathcal{H}^t_\delta(\Lambda)\right\}, \tag{7}$$

where

$$\mathcal{H}^t_\delta(\Lambda) = \inf\left\{\sum_{i=1}^{\infty} |A_i|^t \colon \Lambda \subset \bigcup_{i=1}^{\infty} A_i; |A_i| < \delta\right\}. \tag{8}$$

Then $\mathcal{H}^t$ *is a* metric outer measure*. The* t-dimensional Hausdorff measure *is the restriction of* $\mathcal{H}^t$ *to the* σ*-field of* $\mathcal{H}^t$*-measurable sets which include the Borel sets.*

Let $\Lambda \subset \mathbb{R}^d$ and $0 \leq \alpha < \beta$. Then

$$\mathcal{H}^\beta_\delta(\Lambda) \leq \delta^{\beta-\alpha}\mathcal{H}^\alpha_\delta(\Lambda).$$

Using that $\mathcal{H}^t(\Lambda) = \lim_{\delta\to 0} \mathcal{H}^t_\delta(\Lambda)$:

$$\mathcal{H}^\alpha(\Lambda) < \infty \Rightarrow \mathcal{H}^\beta(\Lambda) = 0 \text{ for all } \alpha < \beta.$$

$$0 < \mathcal{H}^\beta(\Lambda) \Rightarrow \mathcal{H}^\alpha(\Lambda) = \infty \text{ for all } \alpha < \beta.$$

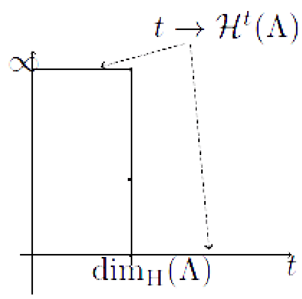


Figure 1: Hausdorff dimension [14]

Definition 6. *The* ***Hausdorff dimension*** *of* Λ *is*

$$dim_H(\Lambda) = \inf\{t \colon \mathcal{H}^t(\Lambda) = 0\} = \sup\{t \colon \mathcal{H}^t(\Lambda) = \infty\}.$$

Definition 7. *In all cases the solution of the equation*

$$r_1^s + \cdots + r_m^s = 1$$

is called the similarity dimension *of the self-similar IFS* $\mathcal{S}$.

Lemma 8. *The Hausdorff dimension of a self-similar IFS in $\mathbb{R}^d$ is always less than or equal to the minimum of d and the similarity dimension s,*

$$\dim_H \Lambda \leq \min\{d, s\}.$$

Moreover, $\mathcal{H}^s(\Lambda) < \infty$.

For self-similar sets having the Open Set Condition (see Definition 10(2)), the similarity dimension, the box-dimension and the Hausdorff dimension should be the same. For the verification it is necessary to estimate the Hausdorff dimension.

A measure μ on $\mathbb{R}^d$ is a *mass distribution* if the support of μ is compact and $0 < \mu(\mathbb{R}^d) < \infty$. For example the Lebesgue measure $\mathcal{L}$ is not a mass distribution on $\mathbb{R}$, but the restriction of $\mathcal{L}$ to any compact set is a mass distribution on $\mathbb{R}$.

Lemma 9 (Mass distribution principle)**.** *Let ν be a mass distribution on $\mathbb{R}^d$ such that* $\operatorname{spt}(\nu) \subset E$. *Assume that for some $t > 0$ there exist $c > 0$ and $\delta > 0$ such that*

$$|A| < \delta \Rightarrow \nu(A) < c \cdot |A|^t.$$

Then we have

$$\mathcal{H}^t(E) \geq \frac{\nu(E)}{c} \text{ and } \dim_H(E) \geq t.$$

This lemma is the simplest way to estimate the Hausdorff dimension of a Borel set $E \subset \mathbb{R}^d$.

Novel properties and conditions (SSP, OSC, SOSC, WSP)

Definition 10. *Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ is a contractiong IFS and Λ is its attractor.*

1. ***The Strong Separation Property (SSP)*** *holds for $\mathcal{F}$ if*

$$f_i(\Lambda) \cap f_j(\Lambda) = \emptyset \ \forall i \neq j.$$

2. ***The Open Set Condition (OSC)*** *holds for $\mathcal{F}$ if there exists a non-empty open set $V \subset \mathbb{R}^d$ such that*

   (a) $f_i(V) \subset V$ holds for all $i = 1, \ldots, m$;

   (b) $f_i(V) \cap f_j(V) = \emptyset$ for all $i \neq j$.

The OSC was introducet by P.A.P. Moran in 1946, and became widely known after the work of J. Hutchinson in 1981.

Theorem 11 (Moran, Hutchinson). *Assume that the self-similar iterated function system $S = \{S_1, \ldots, S_m\}$ acts on $\mathbb{R}^d$ and satisfies the OSC. The similarity ratio of S_i is $0 < r_i < 1$, $i = 1, \ldots, m$. Let s be the similarity dimension, that is, $r_1^s + \cdots + r_m^s = 1$. Then for the attractor Λ of the IFS S we have*

$$0 < \mathcal{H}^s(\Lambda) < \infty.$$

Moreover,

$$\dim_H(\Lambda) = \dim_H(\Lambda) = s,$$

where part of the assertion is that the box dimension exists, that is, the lower and upper box dimensions coincide.

Definition 12. *We say, that **The Strong Open Set Condition (SOSC)** holds for the IFS $\mathcal{S}$ if the OSC holds with an open set V such that $V \cap \Lambda \neq \emptyset$. That is,*

1. *$S_i(V) \subset V$ holds for all $i = 1, \ldots, m$;*
2. *$S_i(V) \cap S_j(V) = \emptyset$ for all $i \neq j$;*
3. *$V \cap \Lambda \neq \emptyset$.*

Theorem 13 (Bandt, Graf and Schief). *For a self-similar IFS $\mathcal{S}$ the following are equivalent:*

1. *OSC*
2. *SOSC*
3. *$0 < \mathcal{H}^s(\Lambda)$,*

where s is the similarity dimension.

Definition 14 (WSP). *The IFS satisfies the **Weak Separation Property (WSP)** if there exists an $l \in \mathbb{N}$ such that for any $\boldsymbol{i} \in \Omega^*$ and every $k \geq 1$, every r^k-ball contains at most l distinct points $f_{\boldsymbol{ji}}(0)$ for $\boldsymbol{j} \in \Lambda_k$.*

In particular if $\{S_j\}_{j=1}^N$ is homogeneous IFS of the form

$$S_j(x) = Lx + t_i, \ 0 < L < 1$$

and $\{S_j\}_{j=1}^N$ then $\{S_i\}_{j=1}^N$ satisfies the WSP if and only if

$$S_\sigma(x_0) = S_{\sigma'}(x_0) \ or \ |S_\sigma(x_0) - S_{\sigma'}(x_0)| \geq a\frac{1}{L^n},$$

where $|\sigma|, |\sigma'| = n$.

It was shown that any IFS satisfying the OSC possesses the WSP, however the converse is not true. No similar relations between the finite type condition and the WSP have been established. In [1] Ngai and Wang introduced the notation of finite type IFS.

Theorem 15 (Nguyen [2]). *If the IFS is of finite type, then it possesses the WSP.*

Dimension of a mass distribution

Definition 16 (Local dimension). *Let μ be a Borel probability measure on $\mathbb{R}^d$ and $x \in \operatorname{spt}(\mu)$. We define the local dimension of the measure μ at x by*

$$d_\mu(x) := \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

if the limit exists. Otherwise we take $\liminf$ *and* $\limsup$ *instead of* $\lim$ *and we obtain the lower local dimension $\underline{d}_\mu(x)$ and the upper local dimension $\bar{d}_\mu(x)$ respectively.*

Definition 17. *Consider a probability space $(\Omega, \mathcal{F}, P)$ and a measurable mapping $T\colon \Omega \to \Omega$.*

1. *We say that T is* measure preserving *if $T_*P = P$, where $(T_*P)(H) := P(T^{-1}H)$ for a Borel set H.*
2. *The T-*invariant σ-algebra *is defined by $\mathcal{I}_T = \{F \in \mathcal{F} | T^{-1}(F) = F\}$.*
3. *We say that T is P-*ergodic *if $P(F)$ is either zero or one for all $F \in \mathcal{I}_T$.*
4. *Let X be a random variable. We say that X is T-*invariant *if $X \circ T = X$. [15]*

Theorem 18 (Birkhoff-Khinchin ergodic theorem). *Let $p \geq 1$, let X be a variable with p^{th} moment and let T be measure preserving and ergodic. Then $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} X \circ T^{k-1} = \mathbb{E}X$ almost surely and in $\mathcal{L}^p$.[15]*

Assume that $\mathcal{S}$ is a self-similar IFS satisfying the SSP and μ is an arbitrary ergodic measure on the symbolic space. ν is the push-down measure of μ. It follows from the Birkhoff Ergodic Theorem (Theorem 18) that the local dimension of ν exists and is equal to a constant at ν-almost all points. If μ is a Bernoulli measure, that is, μ is the infinite product measure $\mu = \boldsymbol{p}^{\mathbb{N}}$ for a probability vector $\boldsymbol{p} = (p_1, \ldots, p_m)$, then for ν-almost all x the limit above exists and is given by

$$\nu = \pi_*\boldsymbol{p}^{\mathbb{N}} \text{ for } \nu \text{ a.a. } x\colon d_\nu(x) = \frac{-\sum\limits_{i=1}^{m} p_i \log p_i}{-\sum\limits_{i=1}^{m} p_i \log r_i} =: \frac{h_{\boldsymbol{p}}}{\mathcal{X}_{\mathbf{r}}^{\boldsymbol{p}}}, \tag{9}$$

where $\mathbf{r} = (r_1, \ldots, r_m)$ is the vector of contraction ratios of the maps from the IFS $\mathcal{S} = \{S_i(x) = r_i x + t_i\}_{i=1}^m$. If ν is the natural measure for $\mathcal{S}$ then $d_\nu(x) \equiv s$, s is the *similarity dimension* of $\mathcal{S}$.

Definition 19. *Let μ be a mass distribution. The Hausdorff dimension of μ is defined as*

$$\dim_H(\mu) := \inf\{\dim_H E \colon \mu(E^c) = 0\}.$$

We can compute the Hausdorff dimension of a measure in terms of its lower dimension.

Theorem 20. *Let μ be a mass distribution. Then*

$$\dim_H(\mu) = \inf\{\alpha \colon \underline{d}_\mu(x) \leq \alpha \textit{ for } \mu\textit{-almost all } x\}.$$

If $\mu(E) = 1$ and $d_\mu(x) = \alpha$ holds for all $x \in E$, then $\dim_H(E) = \alpha$.

Corollary 21. *If $\mathcal{S}$ is a self-similar IFS on $\mathbb{R}^d$ satisfying the SSP with contraction ratios $\mathbf{r}$, ν is the invariant measure for $\mathcal{S}$ with a probability vector $\boldsymbol{p}$, then the Hausdorff dimension of ν can be calculated as in Equation (9):*

$$\dim_H(\nu) = \pi_* \boldsymbol{p}^{\mathbb{N}} \textit{ for } \nu \textit{ a.a. } x \colon d_\nu(x) = \frac{-\sum\limits_{i=1}^m p_i \log p_i}{-\sum\limits_{i=1}^m p_i \log r_i} =: \frac{h_{\boldsymbol{p}}}{\mathcal{X}_{\mathbf{r}}^{\boldsymbol{p}}}.$$

There is a frequently used method (called the potential theoretical caracterisation) to estimate the Hausdorff dimension of a measure based on the following lemma:

Lemma 22. *Let μ be a mass distribution. Then*

$$\dim_H(\nu) \geq \sup\{\alpha \colon \int\int |x-y|^{-\alpha} d\nu(x) d\nu(y) < \infty\}.$$

In case of self-similar measures, this inequality becomes an equality.

2 Kenyon's result on the projection of the Sierpiński carpet

In 1997 Richard Kenyon formulated a theorem in [3], where he gave a condition on projected one-dimensional Sierpinski gaskets. He showed that the projection of Λ in any irrational direction has Lebesgue measure 0, and in a rational direction $\frac{p}{q}$ has Hausdorff dimension less than 1, unless $p+q \equiv 0 \mod 3$. In this case the projection has nonempty interior and measure $\frac{1}{q}$.

Theorem 23 (Kenyon-[3]). *The projection of Λ be the Sierpiński carpet, then the orthogonal projection of Λ_θ in direction θ satisfies:*

- *If $\theta \notin \mathbb{Q}$ then $\mathcal{L}(\Lambda_\theta) = 0$*
- *If $\theta \in \mathbb{Q}$ and $\theta = \frac{p}{q}$, $(p,q) = 1$, then*
  - $\dim_H \Lambda_\theta < 1$ *if* $p + q \not\equiv 0 \text{ mod}3$.
  - Λ_θ *has non-empty interior if* $p + q \equiv 0 \text{ mod}3$.

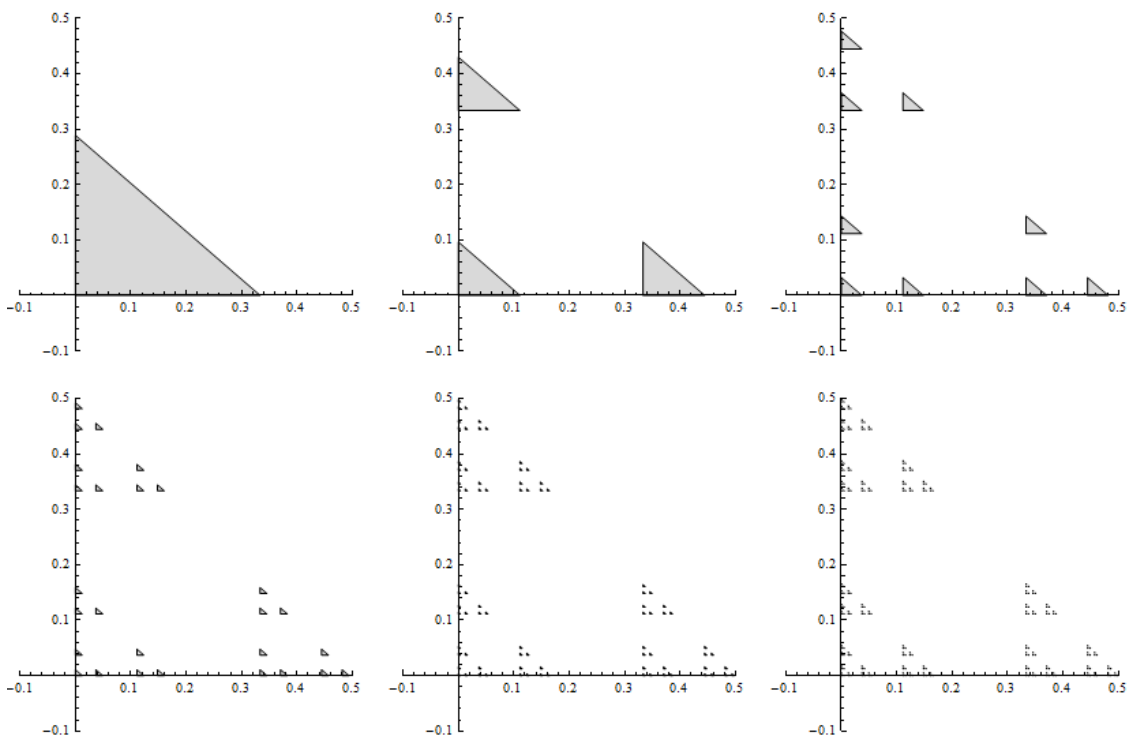


Figure 2: The set Λ defined in (10).

Now we define the right-angled Sierpiński gasket (see Figure 2). Namely let $\Lambda \subset \mathbb{R}^2$ be the attractor of the IFS:

$$\begin{aligned} f_1\colon (x,y) &\mapsto \left(\frac{x}{3}, \frac{y}{3}\right), \\ f_2\colon (x,y) &\mapsto \left(\frac{x+1}{3}, \frac{y}{3}\right), \\ f_3\colon (x,y) &\mapsto \left(\frac{x}{3}, \frac{y+1}{3}\right). \end{aligned} \tag{10}$$

Let Λ_u be the linear projection of Λ onto the x-axis,

$$\Lambda_u = \pi_u(\Lambda), \text{ where } \pi_u = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix}.$$

Observe that the set $\Lambda_u \subset \mathbb{R}$ is the attractor for the three linear maps

$$\begin{aligned} x &\mapsto \frac{x}{3}, \\ x &\mapsto \frac{x+1}{3}, \\ x &\mapsto \frac{x+u}{3}. \end{aligned} \tag{11}$$

In Theorem 23 the measure of Λ_u is computed for every u. This is the first nontrivial example of a dynamically defined set all of whose projected measures can be computed explicitely.

3 Ruiz' method

In this section we review a method introduced by Víctor Ruiz [5]. Although his name is knonwn only by a few, he had great ideas and observations. However, the article[5] where his method was introduced is hard to understand. Hereby I interpret and clearify his results to make them more accessible.

Let μ be a compactly supported finite Borel measure on $\mathbb{R}^d$.

Consider an iterated function system (IFS) $\mathcal{S}\colon \{S_1, \dots, S_n\}$, with contractivity ratio $\frac{1}{L}$, where $L \geq 2,\ L \in \mathbb{Z}$. Let S_i be in the form of

$$S_i(x) = \frac{1}{L}x + \boldsymbol{t}_i, \tag{12}$$

where $\mathbf{t}_i = \mathbf{c}_i(1 - L^{-1})$ is the translation vector with centres $\mathbf{c}_i \in \mathbb{Z}^d$ in a lattice.

We remark that the assumption $\mathbf{t}_i = \mathbf{c}_i(1-L^{-1})$ can be replaced with the assumption that $\mathbf{t}_i$s are selected from an arbitrary lattice on $\mathbb{R}^d$. This follows immediately from the form of the natural projection Π defined in Equation (3).

Example 1. *The IFS from the previous section in Equation (11) can be expressed with the above formula by choosing $d = 1,\ n = 3,\ t_1 = 0,\ t_2 = \frac{1}{3}$ and $t_3 = \frac{u}{3}$, where u satisfies the above conditions, $\mu \ll \mathcal{L}$ for $w_i = \frac{1}{3},\ i \in \{1, 2, 3\}$.*

Let μ be the *self-similar measure* associated to the weighted system of contractive similarities, that is the unique Borel probability measure with

$$\mu = \sum_{i=1}^{n} w_i \cdot \mu \circ S_i^{-1}, \tag{13}$$

where $\{w_1, \dots, w_n\}$ are the weights with the properties $w_i > 0,\ \sum\limits_{i=1}^{n} w_i = 1$.

Since μ is a homogeneus rational self-similar measure, it includes cases with the open set condition (OSC), for which $\dim\mu$ is known. We are interested in those cases where this condition is not satisfied, since our aim is to calculate $\dim\mu$.

For example the Sierpiński gasket can be written in the following form:

- $S_1(x) = \frac{1}{2}\mathbf{x}$
- $S_2(x) = \frac{1}{2}\mathbf{x} + (\frac{1}{2}, 0)^\intercal$
- $S_3(x) = \frac{1}{2}\mathbf{x} + (\frac{1}{4}, \frac{\sqrt{3}}{4})^\intercal$,

where $\boldsymbol{t}_1 = (0, 0)^\intercal,\ \boldsymbol{t}_2 = (\frac{1}{2}, 0)^\intercal,\ \boldsymbol{t}_3 = (\frac{1}{4}, \frac{\sqrt{3}}{4})^\intercal$, and $L = 2$.

In my thesis I only consider one dimensional systems. For this, in the above case, we take the natural projection of the IFS, getting the following $\mathcal{S}$:

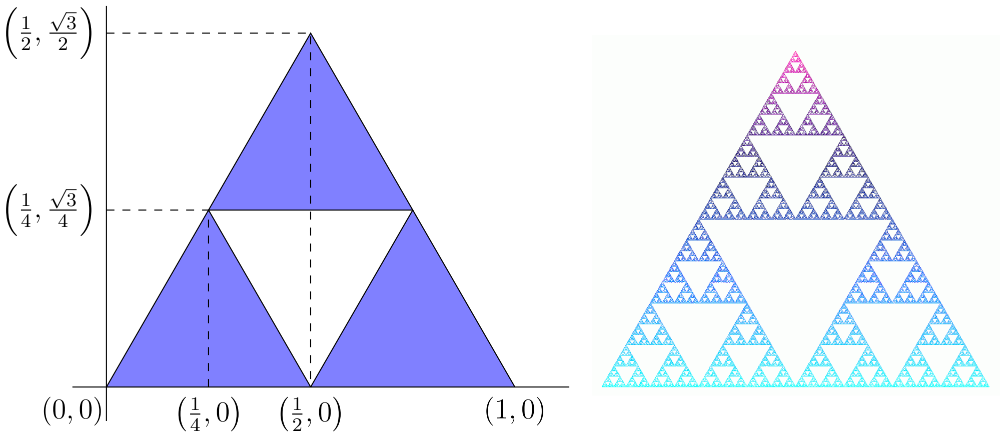


Figure 3: Sierpiński gasket

- $S_1(x) = \frac{1}{2}x$,
- $S_2(x) = \frac{1}{2}x + \frac{1}{2}$,
- $S_3(x) = \frac{1}{2}x + \frac{1}{4}$

with wights $w_1 = w_2 = w_3 = \frac{1}{3}$.

3.1 Matrix expression of the IFS

Preposition

Consider the k^{th} iteration. $\forall k = 0, 1, \ldots$ let $J_k = \{[(i-1) \cdot L^{-k}, i \cdot L^{-k}] \colon i \in \mathbb{Z}\}$ be the class of closed intervals. $\mathcal{D}_k$ is the class of cubes in $\mathbb{R}^d$, the cartesian products of the elements of J_k. By deviding $\mathcal{D}_k$ into L^d parts we get $\mathcal{D}_{k+1}$ with the property $\mathcal{D}_k \subset \mathcal{D}_{k+1}$.

It is immediate that $S_i(\mathcal{D}_{k-1}) = \mathcal{D}_k$ and $S^{-1}(\mathcal{D}_k) = \mathcal{D}_{k-1}$ for all k and i.

The overlaps are nothing else but the union of the boundaries of the cubes in $\mathcal{D}_k$. We need to see that their measure is zero.

Lemma 24 (Ruiz [5]). *Let μ be the self-similar measure associated to a weighted system of contractive similarities $\{(S_i, p_i) \colon i = 1, \ldots, n\} \in \mathbb{R}^d$, possibly having overlaps. If A is not dense in the self-similar set E and $\bigcup_{j=1}^{n} S_j^{-1}(A) \subset A$, then $\mu A = 0$.*

Proposition 25 (Ruiz [5]). *For $k \geq 0$ we have $\mu A_k = 0$.*

There are a finite number of cubes in $\mathcal{D}_0$, since the support of μ is compact. These are $\langle 1 \rangle, \ldots, \langle N \rangle$. Let $M = \{1, \ldots, L^d\}$. Each $\langle j \rangle_{1 \leq j \leq N}$, splits into L^d cubes in $\mathcal{D}_1$, which are $\langle j; i_1\rangle$, $i_1 \in M$. $\langle j; i_1\rangle$ also splits into L^d cubes in $\mathcal{D}_2$. Following the iteration, for $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in M^k$, $\langle j; \boldsymbol{i}\rangle = \langle j; i_1, i_2, \ldots, i_k\rangle$ in the cube $\mathcal{D}_k$. We see that j represents the primary cube.

Now we have the following lemma.

Lemma 26 (Ruiz [5]).

1. *Each $S_l\langle j\rangle$ is a set $\langle i; m\rangle$.*
2. *We have $S_l\langle j\rangle = \langle i; m\rangle$ if and only if $S_l\langle j; \boldsymbol{i}\rangle = \langle i; m, \boldsymbol{i}\rangle$ for all $k \geq 0$ and $\boldsymbol{i} \in M^k$.*
3. *If $S_l^{-1}\langle i; m, \boldsymbol{i}\rangle$ cannot be represented as a set $\langle j; \boldsymbol{i}\rangle$ for any j then $\mu(S_l^{-1}\langle i; m, \boldsymbol{i}\rangle) = 0$.*

Proof.

1. We have $S_l\langle j\rangle = \langle i; m\rangle \in \mathcal{D}_1$ and $\mu(S_l\langle j\rangle) = \sum_{t=1}^{n} w_t \cdot \mu(S_t^{-1} S_l\langle j\rangle) \geq w_l \cdot \mu\langle j\rangle > 0$.
2. It follows from the similarity of S_l.
3. In spite of the first part of Lemma 26, although $S_l^{-1}\langle i; m\rangle \in \mathcal{D}_0$ it can be that it is not a set $\langle j\rangle$. If $S_l^{-1}\langle i; m\rangle \neq \langle j\rangle$ for $j = 1, \ldots, N$, then if it intersects some set $\langle j\rangle$ then it does so in the boundary of it. But we know, that the boundary has the measure 0, so $\mu(S_l^{-1}\langle i; m\rangle) = 0$. From the second part of Lemma 26 it follows that $\mu(S_l^{-1}\langle i; m, \boldsymbol{i}\rangle) = 0$.

□

Construction of the matrices

In the previous subsection we saw that the number of the main cubes are N, and for each $j \in \{1, \ldots N\}$, $\langle j\rangle$ splits into L^d cubes, namely $\langle i; m\rangle$, where $i \in \{1, \ldots, N\}, m \in M = \{1, \ldots, L^d\}$. Remember that the contractivity ratio is L.

For all $m \in M$ let Z_m denote the matrix with dimension $N \times N$ with the following properties:

- $Z_m(i, j) = w_l$ if $S_l\langle j\rangle = \langle i; m\rangle$ for some l and
- $Z_m(i, j) = 0$ otherwise,
- $Z_m(i, j) = w_l$ if $S_l^{-1}\langle i; m, \boldsymbol{i}\rangle = \langle j; \boldsymbol{i}\rangle$.

The last property follows from the the second part of Lemma 26 and the bijectivity of S_l.

We can assume that for all l each S_l are different. If $\exists l, k$ such that $S_l = S_k$ then

$$Z_m(i,j) = w_l + w_k, \text{ if } S_l\langle j\rangle = S_k\langle j\rangle = \langle i;m\rangle.$$

To obtain the formula for computing the matrices Z_m is easy. For $d = 1$ let us consider the closed intervals $I(j) = [j-1, j]$ for $j = 1, \ldots, \max\limits_{1\le l\le n} c_l$. For simplicity we assume that $\min\limits_{1\le l\le n} c_l = 0$. Note that each $\langle i\rangle$ must be a set $I(j)$, but some of the sets $I(j)$ can have null measure and hence not be sets $\langle i\rangle$. We have

$$S_l(x) = \frac{c_l + (x - c_l)}{L}$$

and

$$S_l I(j) = \left[c_l + \frac{j-1-c_l}{L}, c_l + \frac{j-1-c_l}{L} + \frac{1}{L}\right].$$

From this we have

$$\frac{j-1-c_l}{L} = \text{floor}\left(\frac{j-1-c_l}{L}\right) + \text{frac}\left(\frac{j-1-c_l}{L}\right).$$

Now consider, for $j = 1, \ldots, \max\limits_{1\le l\le n} c_l$ and $m = 1, \ldots, L$, the closed intervals $I(j,m) = [j-1+\frac{m-1}{L}, j-1+\frac{m}{L}]$, so that if $I(j) = \langle i\rangle$ then $I(j,m) = \langle i;m\rangle$.

It is easy to check that for given l, j there are a unique i and a unique m with $S_l I(j) = I(i,m)$. Now we obtain $i - 1 = c_l + \text{floor}(\frac{j-1-c_l}{L})$ and $\frac{m-1}{L} = \text{frac}(\frac{j-1-c_l}{L})$, and hence these i, m are

$$i = c_l + 1 + \text{floor}\left(\frac{j-1-c_l}{L}\right),$$

$$m = j - c_l - L \cdot \text{floor}\left(\frac{j-1-c_l}{L}\right).$$

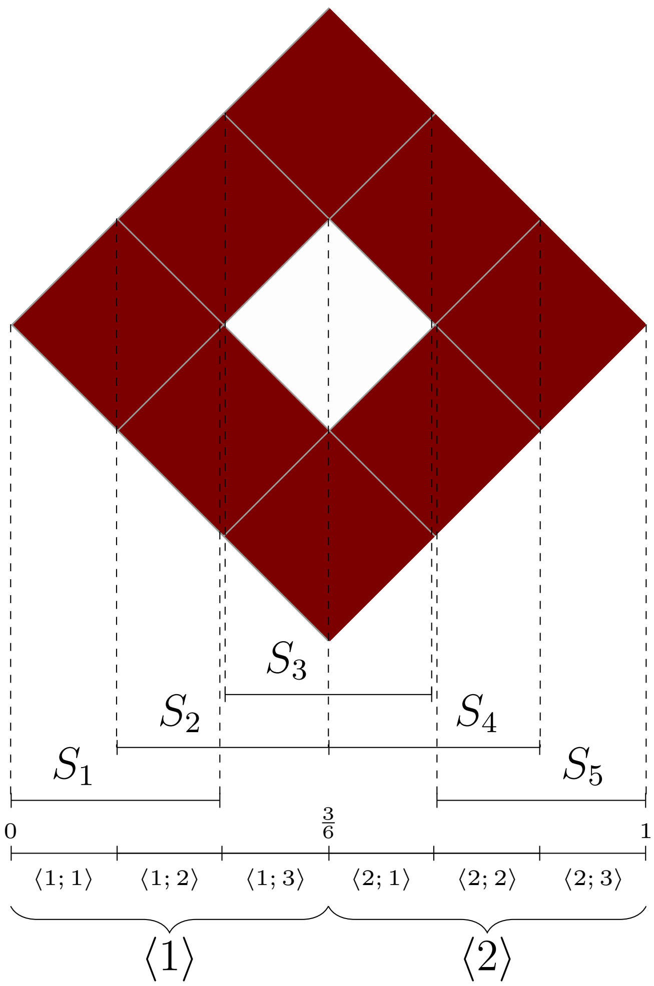
Assume that $\langle i\rangle = I(t_i)$ for $i = 1, \ldots, N$. We have $Z_m(i,j) = w_l$ if $S_l I(t_j) = I(t_i, m)$, and hence we can obtain the matrices Z_m from the two expressions above.

The formula for $d > 1$ can be obtained by considering an expression with d coordinates for m, i, j and $\mathbf{c}_l$.

Examples

The understanding of the matrix construction is not easy, therefore let me provide two examples with explanation below.

Example 2. Rotated Sierpiński carpet with $\mathrm{tg}\alpha = 1$



- $S_1(x) = \frac{1}{3}x$
- $S_2(x) = \frac{1}{3}x + \frac{1}{6}$
- $S_3(x) = \frac{1}{3}x + \frac{2}{6}$
- $S_4(x) = \frac{1}{3}x + \frac{3}{6}$
- $S_5(x) = \frac{1}{3}x + \frac{4}{6}$
- $w_1 = w_5 = \frac{1}{8}$
- $w_2 = w_3 = w_4 = \frac{1}{4}$

Figure 4: Rotated Sierpiński carpet

The matrices are

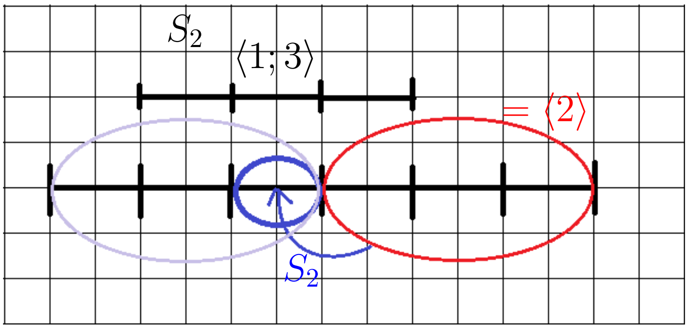
$$Z_1 = \begin{pmatrix} w_1 & 0 \\ w_4 & w_3 \end{pmatrix}, Z_2 = \begin{pmatrix} w_2 & w_1 \\ w_5 & w_4 \end{pmatrix},$$

$$Z_3 = \begin{pmatrix} w_3 & w_2 \\ 0 & w_5 \end{pmatrix}.$$

Explanation

In this case, we have two main cubes, and each cube splits into three parts, since $L = 3$ and $d = 1$. Therefore we will have three matrices.

Consider the case, when $\underbrace{i = 1, j = 2, m = 3}_{Z_3(1,2)=w_2}$.

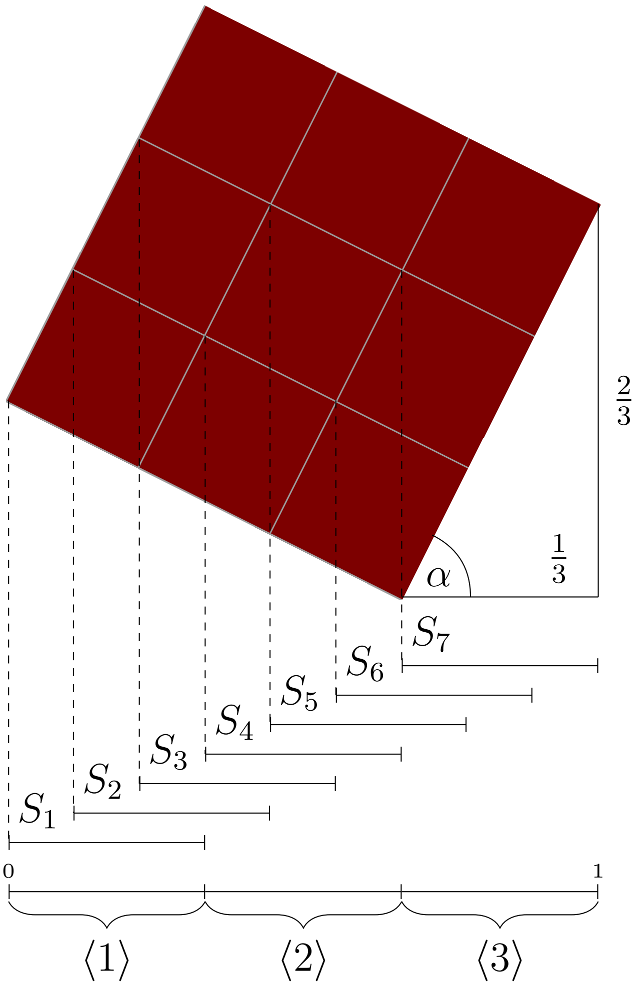


Now we are in the cube $\langle 2 \rangle$, and we want to find the function S_l that projects $\langle 2 \rangle$ to $\langle 1;3 \rangle$. Now we can use the first and second property: $Z_m(i,j) = w_l$ if $S_l\langle j \rangle = \langle i;m \rangle$ for some l and $Z_m(i,j) = 0$ otherwise. Thus the l we are looking for is $l = 2$.

We can see that S_3 also covers $\langle 1;3 \rangle$, but if we devide $S_3[1,6]$ into 2, we get $\langle 1 \rangle$ 'above' $\langle 1;3 \rangle$ instead of $\langle 2 \rangle$.

We get $Z_3(1,2) = w_2 = \frac{1}{4}$.

Example 3. Example for $L = 3, d = 1, N = 3, \mathrm{tg}\alpha = 2$



$$Z_1 = \begin{pmatrix} w_1 & 0 & 0 \\ w_4 & w_3 & w_2 \\ w_7 & w_6 & w_5 \end{pmatrix}$$

$$Z_2 = \begin{pmatrix} w_2 & w_1 & 0 \\ w_5 & w_4 & w_3 \\ 0 & w_7 & w_6 \end{pmatrix}$$

$$Z_3 = \begin{pmatrix} w_3 & w_2 & w_1 \\ w_6 & w_5 & w_4 \\ 0 & 0 & w_7 \end{pmatrix}$$

The functions S_i are the followings:

- $S_1(x) = \frac{1}{3}x$
- $S_2(x) = \frac{1}{3}x + \frac{1}{9}$
- $S_3(x) = \frac{1}{3}x + \frac{2}{9}$
- $S_4(x) = \frac{1}{3}x + \frac{3}{9}$
- $S_5(x) = \frac{1}{3}x + \frac{4}{9}$
- $S_6(x) = \frac{1}{3}x + \frac{5}{9}$
- $S_7(x) = \frac{1}{3}x + \frac{6}{9}$

In case of the projection of the original Sierpiński carpet, the weights w_i are

$$\begin{aligned} w_1 &= w_2 = w_6 = w_7 = \frac{1}{8} \\ w_4 &= 0 \\ w_3 &= w_5 = \frac{2}{8}. \end{aligned}$$

3.2 Measure of the basic cubes

In Subsection 3.1 I introduced the basic cubes and gave a definition of the matrices. Now I would like to continue with the calculation of the basic cubes' measure.

Since $Z_m(i,j) = w_l$, if $S_l^{-1}\langle i; m, \boldsymbol{i}\rangle = \langle j; \boldsymbol{i}\rangle$,

$$w_l \cdot \mu(S_l^{-1}\langle i; m, \boldsymbol{i}\rangle) = Z_m(i,j) \cdot \mu\langle j; \boldsymbol{i}\rangle.$$

From Lemma 26(3)

$$\mu\langle i; m, \boldsymbol{i}\rangle = \sum_{l=1}^{n} w_l \cdot \mu(S_l^{-1}\langle i; m, \boldsymbol{i}\rangle) = \sum_{j=1}^{N} Z_m(i,j) \cdot \mu\langle j; \boldsymbol{i}\rangle. \tag{14}$$

Let $\boldsymbol{e}_j = (0, \ldots, 0, \underbrace{1}_{j}, 0, \ldots, 0)$ the row N-vector with 1 in the jth entry and zero elsewhere.

Let $\mu\langle \cdot; \boldsymbol{i}\rangle = (\mu\langle j; \boldsymbol{i}\rangle : j = 1, \ldots, N)^\intercal$ for $\boldsymbol{i} \in M^k$, so that

$$\mu\langle j; \boldsymbol{i}\rangle = \boldsymbol{e}_j \mu\langle \cdot; \boldsymbol{i}\rangle. \tag{15}$$

The equation (14) can be expressed as

$$\mu\langle \cdot; m, \boldsymbol{i}\rangle = Z_m \mu\langle \cdot; \boldsymbol{i}\rangle. \tag{16}$$

Let

$$\boldsymbol{p}^\intercal = \mu\langle \cdot \rangle = (\mu\langle 1\rangle, \ldots, \mu\langle N\rangle)^\intercal.$$

From (16) we get

$$\mu\langle \cdot; \boldsymbol{i}\rangle = Z_{\boldsymbol{i}} \mu\langle \cdot \rangle = Z_{\boldsymbol{i}} \boldsymbol{p}^\intercal.$$

Using (15) we can obtain

$$\mu\langle j; \boldsymbol{i}\rangle = \boldsymbol{e}_j Z_{\boldsymbol{i}} \boldsymbol{p}^\intercal, \tag{17}$$

if $j = 1, \ldots, N, k \geq 0, \boldsymbol{i} \in M^k$, where $Z_{\boldsymbol{i}} = Z_{i_1} \cdots Z_{i_k}$ if $\boldsymbol{i} = (i_1, \ldots, i_k)$.

The above expression is a very important and useful result. This formula can be used to decide whether an IFS is absolute continuous or not. Moreover, it is easy to calculate. We have already seen the construction of the matrices, and in the followings the method to calculate $\boldsymbol{p}$ will be detailed.

Matrix properties

In this subsection, the propositions are quite important, hence I would like to show their proofs, which can be found in [5].

Proposition 27 (Ruiz [5])**.**

$$Z = \sum_{m \in M} Z_m, \ M = \{1, \dots, L^d\}.$$

Then

1. *Z is irreducible,*
2. *its transpose is stochastic,*
3. *its greatest eigenvalue is* 1 *and it is simple.*

Proof. **Irreducibility:**

Lemma 28 (Ruiz [5])**.** *Let μ be a self-similar measure. If D is an open set, and $\mu(D) \geq 0$, then μ a.e. x $\exists k \geq 1$ and $l_1, \dots, l_k \in \{1, \dots, n\}$, with $S_{l_k}^{-1} \circ \dots \circ S_{l_1}^{-1}(x) \in D$. [16]*

From Proposition 25 it follows that $\mathrm{int}\langle i\rangle$ and $\mathrm{int}\langle j\rangle$ are open sets with positive measures. By Lemma 28 $\exists x \in \mathrm{int}\langle i\rangle$ such that $S_{l_k}^{-1} \circ \dots \circ S_{l_1}^{-1}(x) \in \langle j\rangle$.

$S_{l_1} \circ \dots \circ S_{l_k}\langle j\rangle = \langle t; i_1, \dots, i_k\rangle \ni x \Rightarrow t = i$:

$$S_{l_1} \circ \dots \circ S_{l_k}\langle j\rangle = \langle i; i_1, \dots, i_k\rangle. \tag{18}$$

$t_0 = j, t_k = i$. From Equation (18) and Lemma 26(1), (2) we have that $\exists t_1, \dots, t_{k-1}$, $S_{l_k}\langle t_0\rangle = \langle t_1; t_k\rangle$ and

$$\begin{aligned} S_{l_{k-u}} \circ S_{l_{k-u+1}} \circ \dots \circ S_{l_k}\langle j\rangle &= S_{l_{k-u}}\langle t_u; i_{k-u+1}, \dots, i_k\rangle \\ &= \langle t_{u+1}; i_{k-u}, i_{k-u+1}, \dots, i_k\rangle \end{aligned}$$

where $u = 1, \dots, k-1$. From here with Lemma 26(2) it follows that $Z_{i_{k-u}}(t_{u+1}, t_u) = w_{l_{k-u}} > 0, u = 0, \dots, k-1$.

Hence, for $(i, j) = (t_k, t_0)$ we have

$$Z^k(i, j) \geq Z_{i_1}(t_k, t_{k-1}) Z_{i_2}(t_{k-1}, t_{k-2}) \dots Z_{i_k}(t_1, t_0) > 0,$$

so Z is *irreducible*.

Stochasticity:

From Lemma 26(1) $S_l\langle j\rangle = \langle i; m\rangle\ \forall l, j$, so for all columns of $Z := \sum\limits_{i=1}^{n} w_i = 1$. Therefore the transpose of Z is a stochastic matrix.

Eigenvalue property:

We use the Perron-Frobenius theorem, which completes the proof. $\square$

Proposition 29 (Ruiz [5]). *The unique probability vector* $\mathbf{x}$ *solving the equation* $Z\mathbf{x} = \mathbf{x}$ *is* $\boldsymbol{p}^\intercal$.

Proof. By Proposition 25 $\mu A_0 = 0$, hence $\mu(\langle i\rangle) = 0$. Since $E = \text{supp}\mu \subset \bigcup\limits_{i=1}^{N}$ and $\mu E = 1$ it follows that $\boldsymbol{p}^\intercal$ is a probability vector.

Since $\mu A_1 = 0$ and from Equation 16

$$Z\boldsymbol{p}^\intercal = \sum_{m\in M} Z_m \mu\langle\cdot\rangle = \sum_{m\in M} Z_m \mu\langle\cdot; m\rangle = \mu\langle\cdot\rangle = \boldsymbol{p}^\intercal.$$

Because of the previous proposition, 1 is a simple eigenvalue of Z which proves the uniqueness. $\square$

As I mentioned in the end of the previous subsection, this result gives us an explicit calculation for $\boldsymbol{p}^\intercal$ and then for the $\mu\langle j; \boldsymbol{i}\rangle$. It also can be used to identify the sets $\langle i\rangle$.

Dimension of the self-similar measure

Let η be the auxiliary measure that is constructed by considering the restrictions of μ to the cubes $\langle j\rangle$, translating them to a given fixed cube and piling the restrictions up together. Visually, it is like taking the Sierpiński gasket, and putting the right half on top of the left half of it as is shown in Figure 5.

Let $\langle 0\rangle = [0,1]^d$ the unit cube, and g_j a translation function with $g_j\langle 0\rangle = \langle j\rangle$ for $j = 1, \ldots, N$. Therefore for $\boldsymbol{i} \in M^k$ $g_j^{-1}\langle j; \boldsymbol{i}\rangle = \langle 0; \boldsymbol{i}\rangle$.

$\eta = \sum\limits_{j=1}^{N} \eta_j$ is a Borel measure with $\eta_j = \mu_j \circ g_j$, and $\mu_j(\cdot) = \mu(\langle j\rangle \cap \cdot)$. Furthermore, since μ is a probability measure and the overlaps of the sets $\langle j\rangle$ have null μ-measure,

$$\eta\langle 0; \boldsymbol{i}\rangle = \sum_{j=1}^{N} \mu\langle j; \boldsymbol{i}\rangle,$$

and hence η is also a probability measure.

$\dim\eta = \alpha \Rightarrow \dim\mu = \alpha$ Let Q be the measure on the product σ-algebra on M^∞ given by

$$Q[\boldsymbol{i}] = \boldsymbol{e} Z_{\boldsymbol{i}} \boldsymbol{p}^\intercal,\ \boldsymbol{i} \in M^k,\ k \geq 1,\ \boldsymbol{e} = \sum \boldsymbol{e}_j. \tag{19}$$

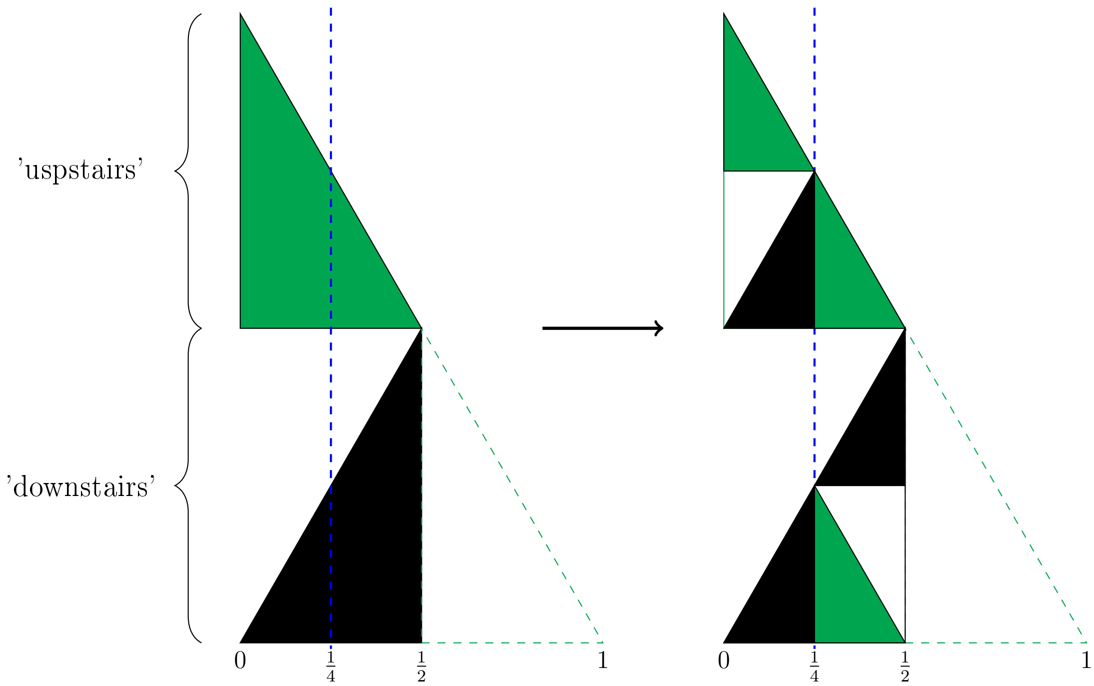


Figure 5: Piling up the restrictions [12]

Q is the distribution of an ergodic hidden Markov chain, which is denoted by

$$V = V(Z_m : m = 1, \dots, L^d) = (V_1, V_2, \dots).$$

By the Shannon-McMillan-Breiman Theorem $\lim_{k\to\infty} -\frac{1}{k}\log_2 Q[i_1, \dots, i_k] = H$.

Let $D_k(x)$ denote the set $\langle 0; i_1, \dots, i_k\rangle$, which containes x.

Proposition 30 (Ruiz [5])**.**

$$\lim_{k\to\infty} -\frac{1}{k}\log_2 \eta D_k(x) = H \ \forall \eta \ a.e.\ x \in \langle 0\rangle.$$

Theorem 31 (Ruiz [5])**.** *Let μ be a homogeneous rational self-similar measure as in Equation (12). Let $V = V(Z_m : m = 1, \dots, L^d)$ be the associated ergodic hidden Markov chain as above, and let H be its Shannon entropy. Then μ is an exact dimensional measure with*

$$\dim\mu = \frac{H}{\log_2 L}.$$

Absolute continuity and singularity

Ruiz obtained the following important results on absolute continuity (Proposition 32) for homogeneous rational self-similar measures, using the above theorem and the result on Shannon entropy: $H(V_1, \dots, V_k) \le \log_2(L^{kd})$ with equality if and only if $\mathbb{P}\{(V_1, \dots, V_k) = \boldsymbol{i}\} = L^{-kd}\ \forall \boldsymbol{i} \in M^k$, where $\mathbb{P}\{(V_1, \dots, V_k) = \boldsymbol{i}\} = Q[\boldsymbol{i}] = \boldsymbol{e}Z_{\boldsymbol{i}}\boldsymbol{p}^\intercal$.

Proposition 32 (Ruiz [5]). *Equivalent properties:*

1. $\mu \ll \mathcal{L}$
2. $\dim \mu = d$
3. $\boldsymbol{e} Z_{\boldsymbol{i}} \boldsymbol{p}^{\intercal} = L^{-kd}$ $\forall k, \boldsymbol{i} \in M^k$
4. η *is the Lebesgue measure on* $[0,1]^d$
5. μ *is not singular.*

Sufficient conditions for absolute continuity

Corollary 33 (Ruiz [5]). *If* $\boldsymbol{e} Z_m = L^{-d} \boldsymbol{e}$ *for all* $m \in M$, *or* $Z_m \boldsymbol{p}^{\intercal} = L^{-d} \boldsymbol{p}^{\intercal}$ *for all* $m \in M$, *then* $\mu \ll \mathcal{L}$.

Proof.

- If $\boldsymbol{e} Z_m = (L^d)^{-1} \boldsymbol{e}$ for all $m \in M$ then $\boldsymbol{e} Z_{\boldsymbol{i}} = (L^d)^{-k} \boldsymbol{e}$ and hence $\boldsymbol{e} Z_{\boldsymbol{i}} \boldsymbol{p}^{\intercal} = (L^d)^{-k} \boldsymbol{e} \boldsymbol{p}^{\intercal} = L^{-kd}$ for all k and $\boldsymbol{i} \in M^k$. Then Proposition 32(3) holds.
- If $Z_m \boldsymbol{p}^{\intercal} = L^{-d} \boldsymbol{p}^{\intercal}$ for all $m \in M$ then $Z_{\boldsymbol{i}} \boldsymbol{p}^{\intercal} = (L^d)^{-k} \boldsymbol{p}^{\intercal}$ and hence $\boldsymbol{e} Z_{\boldsymbol{i}} \boldsymbol{p}^{\intercal} = (L^d)^{-k} \boldsymbol{e} \boldsymbol{p}^{\intercal} = L^{-kd}$ for all k and $\boldsymbol{i} \in M^k$. Then Proposition 32(3) holds.

$\square$

Corollary 34 (Ruiz [5]). *Let* $S_j = \sum \{ w_l \colon \in \{1, \dots, n\}, \mathbf{c}_l \in J(L, \boldsymbol{j}) \}$. *If* $S_{\boldsymbol{j}} = L^{-d}$ *for all* $\boldsymbol{j} \in I$ *then* $\mu \ll \mathcal{L}$.

4 Lau, Ngai, Rau's matrix representation

In [6] a different matrix expression was defined to decide weather an invariant measure for an iterated function system in a form of Equation (12) is absolute continuous or singular. However, the matrix introduced by Víctor Ruiz is different from the one defined in this section, in the future I will try to give a bijection between the two.

4.1 Corresponding theorems from [6]

In the following below I list some theorems corresponding to the singularity of $\mathcal{S}$.

Let Σ_n denote the set of multi-indices $\sigma = (j_1, \dots, j_n)$, $|\sigma| = n$ be the length of σ, and $S_\sigma = S_{j_1} \circ \cdots \circ S_{j_n}$.

In [6] it is shown that the WSC holds for the iterated function system $S_j = \frac{1}{L}x + t_j$, $L \geq 2$ integer. Let $t_j = cr_j$ with $c \in \mathbb{R}$, $r_j \in \mathbb{Q}$. Setting $x_0 = 0$ and $\sigma = (j_1, \dots, j_n)$,

$$S_\sigma(x_0) = S_\sigma(0) = c\sum_{i=1}^{n} \frac{r_{j_i}}{L^{i-1}} = \frac{c}{q}\sum_{i=1}^{n} \frac{b_{j_i}}{L^{i-1}},$$

where $b_j = qr_j$, for $1 \leq j \leq N$, are integers. By taking $a = \frac{c}{q}$ in the second formula in Definition 14, it is seen that the weak separation property is satisfied.

Let μ be the self-similar measure as defined in Equation (13) and $\mathcal{S} = \{S_j\}_{j=1}^N$ be the IFS with associated weights $\{w_j\}_{j=1}^N$. Let

$$\Sigma = \{\bar{\sigma} = (j_1, j_2, \dots) \colon j_i \in \{1, \dots, N\}\}$$

be the trajectory and let Σ_n be the set of σ with length n. Let Π be the natural projection of Σ to $\mathbb{R}^d$ defined by the formula in Equation (3).

Theorem 35 (Lau, Ngai, Rau [6]). *Let $\mathcal{S}$ be an IFS on $\mathbb{R}^d$ as in Equation (12) and assume that it satisfies the WSC. Suppose that $w_j > \frac{1}{L^d}$ for at least one j. Then the self-similar measure is singular.*

This theorem has an interesting consequence corresponding to the density function of μ.

Theorem 36 (Lau, Ngai, Rau [6]). *Suppose that $\mathcal{S}$ satisfies the WSC. If μ is absolute continuous, then the density function $f = D\mu$ will be* bounded*; that is, $f \in L^\infty(\mathbb{R}^d)$. Moreover, f satisfies*

$$f(x) = \sum_{j=1}^{N} w_j f \circ S_j^{-1}(x), \quad x \in \mathbb{R}^d.$$

Corollary 37 (Lau, Ngai, Rau [6]). *Suppose that $\mathcal{S}$ satisfies the WSC. Then the self-similar measure μ is absolutely continuous if and only if the L^2-density of μ exists.*

Definition 38 (Lau, Ngai, Rau [6]). *Let $S_j(x) = A_j(x + d_j)$, where $j = 1, \ldots N$ with $A_j = \rho R_j$. Let $\tilde{C} = (\frac{2\rho}{1-\rho}) \max_j |d_j|$, and let*

$$\tilde{\mathscr{S}} = \{S \in \mathscr{S} \colon |S(0)| \leq \tilde{C}\}.$$

We say that $\{S_j\}_{j=1}^N$ satisfies the weak separation condition* *(WSC*) if $\tilde{\mathscr{S}}$ is a finite set.*

4.2 The new matrix expression

Let $\mathscr{S}$ denote the set of maps $S = S_\sigma^{-1} S_{\sigma'}$ for $(\sigma, \sigma') \in \bigcup_{n=1}^{\infty} (\Sigma_n \times \Sigma_n)$. $\mathscr{S}$ will be considered as a state space and define an (infinete) transition matrix on $\mathscr{S}$ as follows. For $S \in \mathscr{S}$, let

$$T(S) = \sum_{S' \in \mathscr{S}} w_{(S,S')} S'$$

where

$$w_{(S,S')} = \sum_{i,j} \{w_i w_j \colon S_i^{-1} \circ S \circ S_j = S'\}.$$

T can be written as

$$\begin{pmatrix} \tilde{T} & 0 \\ Q & T' \end{pmatrix}$$

where $\tilde{T}$ is a sub-Markov matrix on the states $\tilde{\mathscr{S}}$, since the sum of each column of T is 1, the sum of each column of $\tilde{T}$ is ≤ 1. $\tilde{T}$ is a finite matrix by the WSC*.

Let I be the identity map in $\tilde{\mathscr{S}}$. Let $\mathscr{S}_I$ be the $\tilde{T}$-irreducible component of $\tilde{\mathscr{S}}$ that contains I. Let T_I be the truncated square matrix of $\tilde{T}$ on $\mathscr{S}_I$; then T_I is irreducible and is a finite matrix by the WSC*.

Theorem 39 (Lau, Ngai, Rau [6]). *Suppose that $\mathcal{S}$ is an IFS on $\mathbb{R}^d$ as in Equation (12) and satisfies the WSC*. Then μ is absolutely continuous if and only if*

$$\lambda_{\max} = q^d$$

where $\lambda_{\max}$ is the maximal eigenvalue of T_I.

For the special case in Section 3 let $\mathcal{S}$ be an iterated function system on $\mathbb{R}$ with $S_j(x) = \frac{1}{L}x + t_j$, where $0 < \frac{1}{L} < 1$. Without loss of generality we assume that

$0 = t_1 < t_2 < \cdots < t_N$. By the induction that the state $S = S_\sigma^{-1} S_{\sigma'} \in \mathscr{S}$ has the form

$$S(x) = x + s, \;\; x \in \mathbb{R}$$

for some $s \in \mathbb{R}$. The map S can be represented by the translation number s. The set $\mathscr{S}$ can be constructed inductively, starting from $s = 0$, by letting

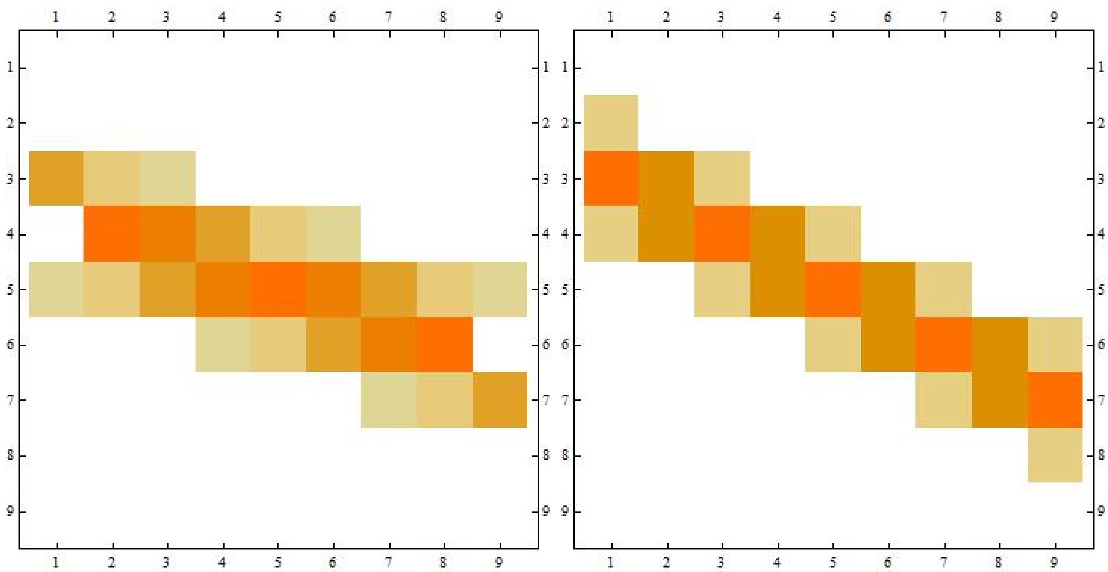
$$s' = L(s + t_j - t_i), \quad 1 \leq i, j \leq N.$$

The set $\tilde{\mathscr{S}}$ can be obtained by keeping those s' with $|s'| \leq \tilde{C} = \frac{\frac{2}{L}}{1-\frac{1}{L}} L * t_N$. The matrix T will send s into the states s' with weight

$$w_{s,s'} = \sum \{w_i w_j \colon L(s + t_j - t_i) = s'\}.$$

According to [6] the above defined $\mathscr{S}$ has the WSC*. In this case $\tilde{T} = T_I$ and $\tilde{T}$ can be reduced further to smaller size by the symmetry of the $\tilde{\mathscr{S}}$.

To see exactly how the matrix construction works in practice, I constructed the matrices for the IFS' showed in Example 2 and 3. To see the structure of the corresponding matrices see Figure 6a and Figure 6b, where the white area corresponds to zero elements, same color area corresponds to the same pozitive elements. We can see, that the matrices have a strangely symmetric structure, which can be exploited for further matrix-reductions.



(a) Matrix form of Example 2 (b) Matrix form of Example 3

Figure 6: Matrices of Examples 2 and 3. White area corresponds to zero elements, same color area corresponds to the same pozitive elements.

5 Sierpiński-like carpets

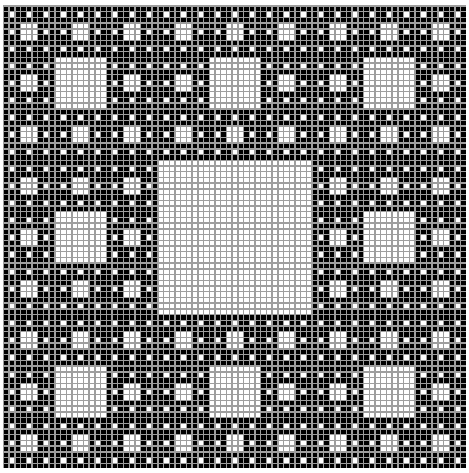
The case of the projected Sierpiński-like carpets is a subset of the set of IFS' in the form defined in Equation (12). In [4] Bárány and Rams showed a sufficient condition that for a fixed rational slope the dimension of almost every intersection w.r.t the nautral measure is strictly greater than $\dim_H \mu - 1$, where μ is the measure of the carpet. They also showed that the dimension of almost every intersection w.r.t the Lebesque measure is strictly less than $\dim_H \mu - 1$, and gave partial multifractal spectra for the Hausdorff and packing dimension of the slices.

Definition 40 (Bárány, Rams [4]). *Let $L \geq 2$ be an integer and let Ω be a subset of $\{0,\dots,L-1\} \times \{0,\dots,L-1\}$. Suppose that $L+1 \leq \sharp\Omega$. Let*

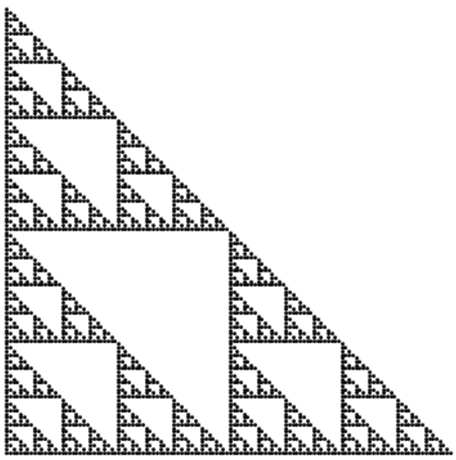
$$S_{k,l}(x,y) := \frac{1}{L}(x,y) + \frac{1}{L}(k,l) \text{ for } (k,l) \in \Omega \tag{20}$$

The attractor $\Lambda \in \mathbb{R}^2$ of the iterated function system $\mathcal{S} = \{S_\omega\}_{\omega\in\Omega}$ is called a ***Sierpiński-like carpet****.*

For example, the usual Sierpiński carpet (Figure 7a) can be expressed as $L = 3$, $\Omega = \{0,1,2\} \times \{0,1,2\} \backslash \{(1,1)\}$, and the usual Sierpiński gasket (Figure 7b), similar to the carpet, has the notation $L = 2$, $\Omega = \{0,1\} \times \{0,1\} \backslash \{(1,1)\}$.



(a) 3×3 Sierpiński carpet



(b) usual Sierpiński gasket

Figure 7: Sierpiński-like carpets

The main purpose of the paper [4] is to investigate the dimension theory of the slices with fixed slope. For an angle θ denote proj_θ the θ-angle projection onto the y-axis. Hence, $\mathrm{proj}_\theta(x,y) = y - x\tan\theta$, and for a point $a \in \mathrm{proj}_\theta\Lambda$ let

$$L_{\theta,a} := \{(x,y) \in \mathbb{R}^d \colon a = y - x\tan\theta\} \text{ and } E_{\theta,a} = L_{\theta,a} \subset \Lambda$$

be the corresponding slice of the attractor. Without loss of generality we can assume that $\theta \in [0, \frac{\pi}{2})$ by applying rotation and mirroring transformations on Λ.

For some notations, let ν be the unique self-similar measure satisfying

$$\nu = \sum_{\omega \in \Omega} \frac{1}{\sharp\Omega} \nu \circ F_\omega^{-1},$$

the natural measure supported on Λ. Furthermore, $\nu_\theta = \nu \circ \mathrm{proj}_\theta^{-1}$ is the projection of the natural measure, $\nu = \frac{\mathcal{H}^s|_\Lambda}{\mathcal{H}^s(\Lambda)}$, where $s = \frac{\log \sharp\Omega}{\log L}$.

For me, the most interesting and relevant result of the article is that under certain conditions the Hausdorff dimension droppes, hence the projected measure is not absolute continuous with respect to the Lebesgues measure.

Proposition 41 (Bárány, Rams [4]). *Let $L \geq 2$ be integer and $\Omega \subset \{0, \dots, L-1\} \times \{0, \dots, L-1\}$ then for every fixed $\theta \in [0, \frac{\pi}{2})$*

$$\dim_H E_{\theta,a} = \dim_B E_{\theta,a} = \frac{\log \sharp\Omega}{\log L} - \dim_H \nu_\theta \text{ for } \nu_\theta\text{-a.e } a.$$

In particular,

$$\dim_H E_{\theta,a} = \dim_B E_{\theta,a} > \frac{\log \sharp\Omega}{\log L} - 1 \text{ for } \nu_\theta\text{-a.e } a. \Leftrightarrow \dim_H \nu_\theta < 1,$$

when $L+1 \leq \sharp\Omega$. In [4] they prove that in the case of rational slopes the stricct inequality is satisfied whenever $L \nmid \sharp\Omega$.

This topic is also discussed in [8] and [7]. Proposition 41 is an extension of the results in these two articles.

In [8, Theorem 9], Manning and Simon proved that for the usual 3×3 Sierpiński carpet, the inequality

$$\dim_H E_{\theta,a} = \dim_B E_{\theta,a} > \frac{\log 8}{\log 3} - 1$$

holds.

Let the usual Sierpiński gasket

$$\Delta = S_0(\Delta) \cup S_1(\Delta) \cup S_2(\Delta), \text{ where}$$

$$S_0(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right), S_1(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right), S_2(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right).$$

In [7, Theorem 1.4] Bárány, Ferguson and Simon showed a similar result for Δ, that is

- for Lebesgue almost all $a \in \Delta_\theta$

$$\alpha(\theta) := \dim_B E_{\theta,a} = \dim_H E_{\theta,a} < s - 1,$$

- for ν_θ-almost all $a \in \Delta_\theta$

$$\beta(\theta) := \dim_B E_{\theta,a} = \dim_H E_{\theta,a} > s - 1,$$

where $s = \dim_B \Delta = \frac{\log 3}{\log 2}$ and $\Delta_\theta = \mathrm{proj}_\theta \Delta$ is the projection of Δ.

Based on these results, Bárány and Rams[4] proved the following two important theorems:

Theorem 42 (Bárány, Rams [4])**.** *Let $L \geq 2$ be integer and $\Omega \subset \{0, \dots, L-1\} \times \{0, \dots, L-1\}$ such that $L+1 \leq \sharp\Omega$ and $L \nmid \sharp\Omega$. Then for every fixed $\theta \in [0, \frac{\pi}{2})$ such that $\tan\theta \in \mathbb{Q}$ there exists a constant $\alpha(\theta)$ depending only on θ such that*

$$\alpha(\theta) = \dim_H E_{\theta,a} = \dim_B E_{\theta,a} > \frac{\log \sharp\Omega}{\log L} - 1 \text{ for } \nu_\theta\text{-a.e } a.$$

A similar theorem can be formalized for Lebesgue-typical points of the projection.

Theorem 43 (Bárány, Rams [4])**.** *Let $L \geq 2$ be integer and $\Omega \subset \{0, \dots, L-1\} \times \{0, \dots, L-1\}$ such that $L+1 \leq \sharp\Omega$ and $L \nmid \sharp\Omega$. For every fixed $\theta \in [0, \frac{\pi}{2})$ such that $\tan\theta \in \mathbb{Q}$ and $\mathrm{proj}_\theta\Lambda = [-\tan\theta, 1]$ there exists a constant β depending only on θ such that*

$$\beta(\theta) = \dim_H E_{\theta,a} = \dim_B E_{\theta,a} < \frac{\log \sharp\Omega}{\log L} - 1 \text{ for Leb.-a.e } a \in \mathrm{proj}_\theta\Lambda.$$

In the proof of Theorem 42, Bárány and Rams introduced and used a new matrix expression, which structure is similar to the one defined in Section 3. The main difference is that the latter one is more general than the former.

In the rest of this section, by using similar methods as they used in the proof, I will give a proof of Theorem 44. Basically it is a generalization of the result on ν_θ in [4].

Theorem 44. *Let μ be the* self-similar measure *associated to rational weights*

$$w_i = \frac{p_i}{q_i}, \gcd(p_i, q_i) = 1 \text{ for } \in \{1, \dots, n\}$$

with the properties $w_i > 0$, $\sum_{i=1}^{n} w_i = 1$. That is

$$\mu = \sum_{i=1}^{n} w_i \cdot \mu \circ S_i^{-1}.$$

Let us denote

$$R = \operatorname{lcm}(q_1, \ldots, q_n). \tag{21}$$

Let the $\mathcal{S} = \{S_i\}_{i=0}^n$ *be the IFS in the form of*

$$S_i(x) = \frac{1}{L}x + \boldsymbol{t}_i,$$

where $\mathbf{t}_i = \mathbf{c}_i(1 - L^{-1})$ *is the translation vector with centres* $\mathbf{c}_i \in \mathbb{Z}^d$ *in a lattice.*

Let μ *be the measure defined above.*

Then, if $L \nmid R$ *then* $\dim_H \mu < 1$.

Remark 1. *If* $L > R$ *then* $\dim_H \mu < 1$. *If this condition holds, then the IFS can be considered as a projected Sierpiński-like carpet with a proper* θ. *Then with* $\Lambda \subset \mathbb{R}^2$ *and* Ω *from Definition 40, and with the substitution* $R = \sharp\Omega$ *we have* $\dim_H \Lambda = \frac{\log \sharp\Omega}{\log L}$, *which is less then* 1 *if* $\sharp\Omega < L$. *From this, it is obvious that the projected dimension is also less then* 1.

Remark 2. *I proove it for* $d = 1$, *but the results can be extended trivially to higher dimensions.*

Remark 3. *In the proof below we will follow the steps of the proof of Bárány, Rams [4, Theorem 1.2], but it is more convinient for us to use some notation from Ruiz [5]. We use the following substitutions:*

$$\sharp\Omega = R, \;\; N = L, p + q = N.$$

Proof. First let us check that the IFS $\mathcal{S}$ defined in Theorem 44 includes the case of the projected Sierpiński carpets from [4, Theorem 1.2].

Let (x, y) be a point from a Sierpiński-like carpet. Let θ is the angle of the projection onto the y-axis. Then the projected iterated function system $\phi = \{f_\omega\}$ of $\{S_\omega\}_{\omega \in \Omega}$, where S_ω is in the form of Equation (20). Then

$$f_{k,l}(x) = \frac{x}{L} + \frac{-k \tan\theta + l}{L}, \text{ for every } (k, l) \in \Omega.$$

Since $k, l \in \mathbb{Z}$, $\tan\theta = \frac{p}{q}$, $p, q \in \mathbb{Z}$ and $x \in I$, where $I = [-\tan\theta, 1]$. Let us modify I by multiplying with q and translate it with p. Then, for $I' = [0, p + q]$, let

$$f'_{k,l}(x) = \frac{x}{L} + \frac{-kp + lq + Lp}{L}, \text{ for every } (k, l) \in \Omega.$$

From this we can see that the IFS in Equation 12 with translation $t_i = \frac{L \cdot c_i - c_i}{L}$ includes the system $\phi' = \{f'_\omega\}$.

Let us consider the matrices and the notations introduced and used in Section 3. In Proposition 27 we have seen that the matrix $Z^\intercal$ is stochastic, therefore the equation

$$\sum_{i=1}^{N} \sum_{m \in M} (Z_m)_{i,j} = 1 \text{ for every } j = 1, \ldots, N$$

holds. From this,

$$Z^\intercal = \sum_{m \in M} Z_m^\intercal$$

defines a Markov-chain on $\Theta := \{1, \ldots, N\}$.

We consider the closed intervals $I(j) = [j-1, j]$ for $j \in \{1, \ldots, N\}$. Moreover, $I(j,m) = [j-1+\frac{m-1}{L}, j-1+\frac{m}{L}]$, so that if $I(j) = \langle i \rangle$ then $I(j,m) = \langle i; m \rangle$.

Let us devide the set of states into two parts.

$$\Theta_r = \{i \in \Theta \colon \mu(I(i)) > 0\}$$
$$\Theta_t = \{i \in \Theta \colon \mu(I(i)) = 0\}.$$

Note that each $\langle i \rangle$ must be a set $I(j)$, but some of the sets $I(j)$ can have null measure and hence not be sets $\langle i \rangle$ [5].

Lemma 45. *The set Θ_r is a recurrent class and Θ_t is a transient class of the Markov-chain defined by $Z^\intercal$. Moreover, Θ_r is aperiodic.*

Proof. First show that if $i \in \Theta_r$ and $Z_{i,j}^\intercal > 0$ then $j \in \Theta_r$. Since $Z_{i,j}^\intercal > 0$ there exist $k \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, L\}$ such that $S_k(I(i)) = I(j,m)$. Therefore $0 < \mu(S_k(I(i))) = \mu(I(j,m)) \leq \mu(I(j))$.

On the other hand, for every $K > 0$ sufficiently large and for every $j \in \Theta_r$ there exists a vector $\mathbf{k} \in \{1, \ldots, n\}^K$ such that $S_{\mathbf{k}}(I) \subset I(j)$. This implies that for every $j \in \Theta_r$ and every $i \in \Theta$ $((Z^\intercal)^K)_{i,j} > 0$, which proves the statement. $\square$

From the theory of Markov-chains there exists a unique probability vector $\boldsymbol{p}$ such that $\boldsymbol{p}$ is the stationary distribution of $Z^\intercal$, i.e. $\boldsymbol{p}^\intercal Z^\intercal = \boldsymbol{p}^\intercal$. In particular,

$$\left(\sum_{m \in M} Z_m^\intercal \right) \boldsymbol{p} = \boldsymbol{p}.$$

Let us observ, that $\boldsymbol{p}_i = \mu(I_i)$.

From Equation (17) we know that

$$\mu\langle j; \boldsymbol{i} \rangle = \boldsymbol{e}_j Z_{\boldsymbol{i}}^\intercal \boldsymbol{p}^\intercal = Q[\boldsymbol{i}], \ \boldsymbol{i} \in M^k, \ k \geq 1,$$

where $\boldsymbol{e} = \sum \boldsymbol{e}_j$, e_j denotes the i^{th} element of the natural basis of $\mathbb{R}^{p+q}$, and Q is a shift-invariant and ergodic probability measure.[5]

Denote Z_m^r the submatrix of $Z_m^\intercal$ by deleting the rows and columns of Θ_t. If $j \in \Theta_r$ and $i \in \Theta_t$ then $(Z_m)_{i,j} = 0$ for every $m \in M$. Hence,

$$\sum_{i \in \Theta_r} \sum_{m \in M} (Z_m^r)_{i,j} = 1 \tag{22}$$

for every $j \in \Theta_r$.

Lemma 46. *For any $i, j \in \Theta_r$ and $m_1, \ldots, m_n \in M$*

$$(Z_{m_1}^\intercal \ldots Z_{m_n}^\intercal) = (Z_{m_1}^r \ldots Z_{m_n}^r)_{i,j}.$$

Proof. Prove by induction. For $n = 2$

$$(Z_{m_1}^\intercal Z_{m_2}^\intercal)_{i,j} = \sum_{k=1}^{N} (Z_{m_1}^\intercal)_{i,k} (Z_{m_2}^\intercal)_{k,j} = \sum_{k \in \Theta_r} (Z_{m_1}^\intercal)_{i,k} (Z_{m_2}^\intercal)_{k,j} = (Z_{m_1}^r Z_{m_2}^r)_{i,j}.$$

We used in the second equation that $(Z_{m_2})_{k,j} = 0$ whenever $k \in \Theta_t$. Then

$$(Z_{m_1}^\intercal \ldots Z_{m_n}^\intercal Z_{m_{n+1}}^\intercal)_{i,j} = \sum_{k=1}^{N} (Z_{m_1}^\intercal \ldots Z_{m_n}^\intercal)_{i,k} (Z_{m_{n+1}}^\intercal)_{k,j}.$$

Again, $(Z_{m_{n+1}})_{k,j} = 0$ whenever $k \in \Theta_t$, so

$$\sum_{k \in \Theta_r} (Z_{m_1}^\intercal \ldots Z_{m_n}^\intercal)_{i,k} (Z_{m_{n+1}}^\intercal)_{k,j} = (Z_{m_1}^r \ldots Z_{m_{n+1}}^r)_{i,j}.$$

□

An important consequence of the previous lemma is that for every $m_1, \ldots, m_n \in M$ and $i \in \Theta_r$

$$\mu \langle j; \boldsymbol{i} \rangle = \widehat{\boldsymbol{e}}_j Z_{\boldsymbol{i}}^r \widehat{\boldsymbol{p}}^\intercal,$$

where $\widehat{\boldsymbol{p}} = (\mu(I_i))_{i \in \Theta_r}$ and $\widehat{\boldsymbol{e}}_i$ is the ith element of the natural basis of $\mathbb{R}^{\sharp \Theta_r}$.

Now define a left/shift invariant measure κ on the symbolic space $\Sigma = M^{\mathbb{N}} = \{1, \ldots, L\}^{\mathbb{N}}$. Endow Σ with the metric $d(\mathbf{m}, \underline{\zeta}) = L^{-n}$ for $\mathbf{m} = (m_1, m_2, \ldots)$ and $\underline{\zeta} = (\zeta_1, \zeta_2, \ldots)$, where n is the larges integer such that $m_i = \zeta_i (1 \leq i \leq n)$. For a cylinder set $[m_1, \ldots, m_n] = \{(\zeta_1, \zeta_2, \ldots) \in \Sigma \colon \zeta_k = m_k,\ k = 1, \ldots, n\}$ let

$$\kappa([m_1, \ldots, m_n]) := \widehat{\boldsymbol{e}} Z_{\boldsymbol{i}}^r \widehat{\boldsymbol{p}}^\intercal, \tag{23}$$

where $\widehat{\boldsymbol{e}} = \sum_{i \in \Theta_r} \widehat{\boldsymbol{i}}_i$. By Equation (22), κ is a probability measure.

Lemma 47. *The probability measure κ is σ-invariant and mixing and hence ergodic, where σ denotes the left-shift operator on Σ.*

Proof. First, we prove the invariance. It is enough to prove for the cylinder sets. Since the vector $\widehat{\boldsymbol{e}}$ is a left-eigenvector of $\sum_{m=0}^{L} Z_m$ (follows from Equation (22)), then for a cylinder set $[m_1, \ldots, m_n]$

$$\begin{aligned}
\kappa(\sigma^{-1}[m_1, \ldots, m_n]) &= \sum_{m=1}^{L} \kappa([m, m_1, \ldots, m_n]) = \\
&= \sum_{m=0}^{L} \widehat{\boldsymbol{e}}^{\mathsf{T}} Z^r_m Z^r_{m_1} \cdots Z^r_{m_n} \widehat{\boldsymbol{p}} = \widehat{\boldsymbol{e}}^{\mathsf{T}} Z^r_{m_1} \cdots Z^r_{m_n} \widehat{\boldsymbol{p}} \\
&= \kappa([m_1, \ldots, m_n]).
\end{aligned}$$

To prove the mixing property it is enough to show that for any cylinder sets $[m_1, \ldots, m_k]$ and $[\zeta_1, \ldots, \zeta_l]$

$$\lim_{n\to\infty} \kappa([m_1, \ldots, m_k] \cap \sigma^{-1}[\zeta_1, \ldots, \zeta_l]) = \kappa([m_1, \ldots, m_k])\kappa([\zeta_1, \ldots, \zeta_l]).$$

By the definition of κ in Equation (23), for sufficiently large n

$$\begin{gathered}
\kappa([m_1, \ldots, m_k] \cap \sigma^{-n}[\zeta_1, \ldots, \zeta_l]) = \\
\sum_{i_1, \ldots, i_{n-k}=1}^{L} \widehat{\boldsymbol{e}}^{\mathsf{T}} Z^r_{m_1} \cdots Z^r_{m_k} Z^r_{i_1} \cdots Z^r_{i_{n-k}} Z^r_{\zeta 1} \cdots Z^r_{\zeta_l} \widehat{\boldsymbol{p}} = \\
\widehat{\boldsymbol{e}}^{\mathsf{T}} Z^r_{m_1} \cdots Z^r_{m_k} \left(\sum_{i=1}^{L} Z^r_i \right)^{n-k} Z^r_{\zeta 1} \cdots Z^r_{\zeta_l} \widehat{\boldsymbol{p}}.
\end{gathered}$$

Applying Lemma 45 and the basic properties of aperiodic, irreducible Markov chains, we have

$$\lim_{n\to\infty} \left(\sum_{i=1}^{L} Z^r_i \right)^{n-k} = \widehat{\boldsymbol{p}} \widehat{\boldsymbol{e}}^{\mathsf{T}},$$

which implies the mixing property. □

In the proof of the next lemma we will need two well-known theorems:

Theorem 48 ([9, Theorem 4.10])**.** *If $T\colon X \to X$ is measure-preserving and $\mathcal{A}$ is a finite sub-algebra of $\mathcal{B}$ then $\frac{1}{n} H \left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{A} \right)$ decreases to $h(T, \mathcal{A})$.*

Theorem 49 ([9, Theorem 4.18])**.** *If T is a measure-preserving transformation (but not necessarily invertible) of the probability space $(X, \mathcal{B}, m)$ and if $\mathcal{A}$ is a finite sub-algebra of $\mathcal{B}$ with $\bigvee_{i=0}^{\infty} T^{-i}\mathcal{A} \doteq \mathcal{B}$ then $h(T) = h(T, \mathcal{A})$.*

Lemma 50. *Denote by h_κ the entropy of measure κ. If $L \nmid R$ then $h_\kappa < \log L$.*

Proof. We argue by contradiction. Suppose that $h_\kappa = \log L$.

By Theorem 48 and 49 we have that

$$h_\kappa = \lim_{n\to\infty} -\frac{1}{n} \sum_{m_1,\dots,m_n=1}^{L} \widehat{\boldsymbol{e}}^\intercal Z^r_{m_1}\cdots Z^r_{m_n}\widehat{\boldsymbol{p}} \log \widehat{\boldsymbol{e}}^\intercal Z^r_{m_1}\cdots Z^r_{m_n}\widehat{\boldsymbol{p}},$$

and the left hand side decreases as $n \to \infty$. That is, $h_\kappa = \log N$ if and only if

$$\widehat{\boldsymbol{e}}^\intercal Z^r_{m_1}\cdots Z^r_{m_n}\widehat{\boldsymbol{p}} = \frac{1}{L^n}, \tag{24}$$

for every $n \geq 1$ and $m_1, \dots, m_n \in \{1, \dots, L\}$.

By Lemma 45 there exists a $K > 0$ such that $\left(\sum_{m=1}^{L} Z^r_m\right)^K > 0$, because each element of the matrix is strictly positive. Without loss of generality, we may assume that $K > N^2 + 1$. Then there exists a word $(\zeta_1, \dots, \zeta_K)$ of length K such that $\left(\sum_{m=1}^{L} Z^r_m\right)^K - Z^r_{\zeta_1}\cdots Z^r_{\zeta_K} > 0$. Let $\mathcal{A} := \{1, \dots, L\}^K\{(\zeta_1, \dots, \zeta_K)\}$. By Perron-Frobenius theorem there exists a $\rho > 0$ and $\boldsymbol{u}, \mathbf{v}$ vectors such that ρ is the largest eigenvalue of the matrix $\sum_{\mathbf{m}\in\mathcal{A}} Z^r_{\mathbf{m}}$ and $\boldsymbol{u}, \mathbf{v}$ are the corresponding left and right eigenvectors. Moreover,

$$\lim_{n\to\infty} \frac{1}{q^n}\left(\sum_{\mathbf{m}\in\mathcal{A}} Z^r_{\mathbf{m}}\right)^n = \mathbf{v}\boldsymbol{u}^\intercal. \tag{25}$$

By the assumption in Equation (24)

$$\frac{1}{n}\log \widehat{\boldsymbol{e}}^\intercal \left(\sum_{\mathbf{m}\in\mathcal{A}} Z^r_{\mathbf{m}}\right)^n \widehat{\boldsymbol{p}} = \log\frac{\sharp\mathcal{A}}{L^K} = \log\frac{L^K - 1}{N^K}.$$

On the other hand, by Equation (25)

$$\lim_{n\to\infty}\frac{1}{n}\log \widehat{\boldsymbol{e}}^\intercal \left(\sum_{\mathbf{m}\in\mathcal{A}} Z^r_{\mathbf{m}}\right)^n \widehat{\boldsymbol{p}} = \log\rho.$$

So $\rho = 1 - L^{-K}$ but, because of the definition of R in Equation (21), $R\cdot Z^r := R\sum_{\mathbf{m}\in\mathcal{A}} Z^r_{\mathbf{m}} \in \mathbb{Z}^{N\times N}$, hence $R\rho = R - RL^{-K} \in \mathbb{Q}\backslash\mathbb{Z}$. But we arrived to a contradiction because this cannot be a root of characteristic polynomial of $R\cdot Z^r$, which is a matrix of integer coefficients. □

Now we finish the proof of Theorem 44. By using Theorem 31 from Subsection 3.2 we conclude that

$$\dim_H \mu = \frac{h_\kappa}{\log L} < 1.$$

□

Conclusion

In my thesis I focus on iterated function systems construated with overlapping parts. These IFSs are from a special family, namely where the linear part in each map are the same. This linear part is a concrate contraction which can be interpreted as a reciprocal of a natural number, and the translations are chosen from a lattice in $\mathbb{Z}^d$.

This thesis gives a brief overview of the corresponding results on geometric measure theory. During the investigation of the corresponding literature, I studied Richard Kenyon's work [3] on the projection of the Sierpiński carpet. A more recent paper from 2000 by Lau, Ngai and Rau [6] gave a matrix expression which fully represents an IFS. Using matrix representations for such IFSs is quite useful, and better for computations. Thus Víctor Ruiz [5] shows a new method for this purpose, then with the help of the matrices Ruiz investigates the relation between Hausdorff dimensions and absolute continuity. In future work it would be interesting to find a relation between the two types of matrices, hence absolute continuity could be prooven easier and with more efficiency.

The thesis contains a new, original result. Namely, I extend a theorem due to Balázs Bárány and Michal Rams, about the dimension of the projection of self-similar measures of the generalized Sierpinski Gasket, to more general IFSs on the line. I verify that the same method can be applied in this more general settings.

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