1. Determine the number of symmetries (and describe these symmetries) for the following geometrical objects:
  a) a square;
  b) a rectangle which is not a square;
  c) a circle;
  d) the part of the plane enclosed by two parallel lines;
  e) a cube;
  f) a regular tetrahedron.

2. Suppose that a geometrical object in $\mathbb{R}^2$ or $\mathbb{R}^3$ has a symmetry which reverses the orientation. Prove that in this case the given object has the same number of orientation preserving and orientation reversing symmetries.

3. Let $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \}$ be an arbitrary basis of the euclidean space $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation. Prove that f is orthogonal if and only if $f(\mathbf{v}_i) \cdot f(\mathbf{v}_j) = \mathbf{v}_i \cdot \mathbf{v}_j$ for all i, j.

4. Prove that every orthogonal transformation of $\mathbb{R}^2$ is either a rotation about the origin or a reflection through a line containing the origin. What can we say about the orthogonal transformations of $\mathbb{R}^3$?

5. Let H be a set of at least two elements. Prove that the power set $P(H)$ is not a group with respect to the intersection or union operation but it is a group with respect to the symmetric difference (i.e. with the operation $A \triangle B := (A \cup B) \setminus (A \cap B)$).

6. Prove that the axiom of associativity implies that the value of a product $a_1 \cdot a_2 \cdot \dots \cdot a_n$ is the same for any distribution of parantheses.

7. Check whether the following sets form a group with respect to the usual addition or multiplication:
  a) $n \times n$ real matrices with determinant equal to 1;
  b) $n \times n$ real matrices with positive determinant;
  c) $n \times n$ integral matrices;
  d) $n \times n$ integral matrices with nonzero determinant;
  e) $n \times n$ integral matrices with determinant equal to 1;
  f) $n \times n$ upper triangular real matrices.

8. Let K be field and consider the set of all $K \to K$ mappings. Prove that the subset $\{ x \mapsto ax+b \mid a, b \in K,\ a \neq 0 \}$ forms a group with respect to the composition of functions. Find nontrivial subgroups of this group. (Recall that a subgroup is called trivial if it is equal to the one element subgroup or the full group.)

9. Let S be a semigroup with 1. Prove the following:
  a) if a and b are invertible, then both ab and ba are invertible;
  b) if both ab and ba are invertible then a and b are also invertible.

Show that invertibility of ab does not imply the invertibility of a or b by giving an example.

10. a) Prove that the intersection of any system of subgroups in a group must be a subgroup.
  b) Show that the union of two subgroups is a subgroup if and only if one of the subgroups contains the other.