1. Let G be a group and $H \leq G$. Which of the following statements are true?

a) There exists a homomorphism from G with kernel H.
b) If $H \triangleleft G$ then there exists a homomorphism from G with kernel H.
c) There exists a homomorphism to G with image H.
d) For any homomorphism $\varphi : G \to K$, $\varphi(H) \leq K$.
e) For any $H \triangleleft G$ and any homomorphism $\varphi : G \to K$, $\varphi(H) \triangleleft K$.
f) If $H \triangleleft G$ and $\varphi : G \to K$ is a homomorphism then $\varphi(H) \triangleleft \varphi(G)$.
g) For any homomorphism $\varphi : G \to K$, $n \mid |G|$ implies $n \mid |\varphi(G)|$.
h) For any homomorphism $\varphi : G \to K$, $n \mid |\varphi(G)|$ implies $n \mid |G|$.

2. Let $|G| = 91$. Determine the number of homomorphisms $G \to G$ that map at least two nonidentity elements of different order into 1.

3. Let $N \triangleleft G$ and $H \leq G$ such that $H \cap N = \{1\}$ and $HN = G$. Prove that $G/N \cong H$.

4. Let $H \leq G$ and $M, N \triangleleft G$. Prove that

a) $H \cap N \triangleleft H$; $N \cap M \triangleleft G$;
b) $HN \leq G$; $NM \triangleleft G$;
c) $M \leq N$ implies that G/N is a homomorphic image of G/M.

5. Let $N \triangleleft G$. Prove that the map $H \mapsto H/N := \{ Nh \mid h \in H \}$ is a bijection between the subgroups of G containing N and the subgroups of the factor group G/N. Prove further that normal subgroups are mapped to normal subgroups both by the given bijection and its inverse, and that this map preserves the partial order defined by the inclusion of subgroups.

6. Prove that for any normal subgroup N of even index in a group G there exists a subgroup H with $N \leq H \leq G$ and $|H : N| = 2$.

7. Let $g \in G$ and $o(g) = nm$, where $(n, m) = 1$. Prove that g can be written uniquely in the form uv with $u, v \in \langle g \rangle$, $o(u) = m$, and $o(v) = n$.

8. Prove that for any natural numbers n, m with $(n, m) = 1$, the cyclic group C_{nm} is isomorphic to $C_n \times C_m$.

9. a) Let H and K be two groups with $(|H|, |K|) = 1$. Prove that every subgroup of $G = H \times K$ can be written as $H_1 \times K_1$, where $H_1 \leq H$ and $K_1 \leq K$.
b) Find a subgroup of order 4 in $D_4 \times C_4$ such that cannot be decomposed into a direct product of subgroups of the two components.

Hf1. Let $H \leq G$. Prove that $\bigcap_{g \in G} g^{-1}Hg$ is the largest normal subgroup of G which is contained in H.

Hf2. Let A and B be Abelian subgroups of a group G such that $G = AB$. Prove that $A \cap B \triangleleft G$.

Hf3. Let G be a group of order 60, which has a nontrivial homomorphism to a group of order 28. Prove that G ha a subgroup of index 2.