1*. Let R be a finite commutative ring. Prove that R has an identity if and only if $aR \neq \{ 0 \}$ for any $0 \neq a \in R$.

2. Let $1 \in R$ be a ring with no zero divisors. Prove that a right inverse of any element is also its left inverse.

3. What are the (right, left, two-sided) ideals in the following rings?
a) $\mathbb{Z}_n$;
b) $\mathbb{R}^{n \times n}$;
c) $\mathbb{R}[x]/(x^2+1)$;
d) $\mathbb{C}[x]/(x^2+1)$;
e) $n \times n$ upper triangular matrices over $\mathbb{Z}$.

4. Describe the elements of the ideal generated by x and y^2 in the ring $K[x,y]$, where K is a (commutative) field. Give a transversal of the cosets of the ideal. Find the ideals of the factor ring.

5. Let $R = 2\mathbb{Z}$ be the ring of even integers, and $R_1 = \{(a,m) \,|\, a \in R,\ m \in \mathbb{Z}\}$ the usual exension of R to a ring with identity. Prove that R_1 is not isomorphic to $\mathbb{Z}$.

6. Determine the fraction field of $2\mathbb{Z}$.

7. Prove that a ring has no proper right ideal if and only if it is a (skew) field or a zero ring of prime order.

8. Prove that every finite integral domain is a field.

9. Prove that the ideal of $\mathbb{Z}[x]$ generated by 2 and x is not a principal ideal.

10. Prove that $\mathbb{Z}[i] = \{ a + bi \,|\, a, b \in \mathbb{Z} \}$ is a euclidean ring.

HW1. Prove that the nilpotent elements of a commutative ring form an ideal.

HW2. Let R be a commutative ring with identity. Prove that $I, J \lhd R$ and $I + J = R$ implies $IJ = I \cap J$.

HW3. Let $R = \mathbb{Z}[x]$, and let I be the ideal of R generated by 2 and x^2. Prove that the factor ring R/I has 4 elements, and that its multiplicative semigroup is isomorphic to the multiplicative semigroup of $\mathbb{Z}_4$.