1. Which of the following are principal ideal rings?

a) $\mathbb{Z}$ b) $\mathbb{R}$ c) $\mathbb{Z}[x]$ d) $\mathbb{R}[x]$ e) $\mathbb{Z}[i]$

2*. Prove that for an arbitrary field K the ideal $(x_1, \ldots, x_n)$ in the polynomial ring $K[x_1, \ldots, x_n]$ cannot be generated by less than n elements.

3. What are the irreducible and what are the prime elements in the ring of even integers? Determine the ideals and the principal ideals of the ring $2\mathbb{Z}$.

4. Prove that:

a) the elements of $\mathbb{Z}[\sqrt{d}]$ (where $d \in \mathbb{Z}$ is an integer which is not a square of an integer) can be uniquely written in the form $a + b\sqrt{d}$, where $a, b \in \mathbb{Z}$;
b) the norm defined by $N(a + b\sqrt{d}) = a^2 - b^2 d$ is multiplicative;
c) for any $z, u \in \mathbb{Z}[\sqrt{d}]$ we have $z \mid u \Rightarrow N(z) \mid N(u)$;
d) z is a unit in $\mathbb{Z}[\sqrt{d}] \Leftrightarrow N(z) = \pm 1$.

5. Write as a product of prime elements in $\mathbb{Z}[i]$ the Gaussian integers 7, 13 és $5 + i$. How many mutually distinct prime factors does the number $2+2i$ have? (Two Gaussian integers are considered mutually distinct if neither of them can be obtained from the other by multiplying it with a unit.)

6. Let us take $R = \mathbb{Z}[\sqrt{-5}]$. Show that 6 has (at least) two distinct factorizations into a product of irreducible elements.

7. Suppose that $\mathbb{Z}[\sqrt{d}]$ (for some $d \in \mathbb{Z}$ which is not a square of an integer) is a unique factorization domain (UFD). Prove that 2 cannot be irreducible in $\mathbb{Z}[\sqrt{d}]$.

8. Suppose $d \in \mathbb{Z}$ is square-free (i.e. it is not divisible by any square number). Show that:

a) for $d < 0$ the ring $\mathbb{Z}[\sqrt{d}]$ is a UFD $\Leftrightarrow d = -1$ or -2.
b) $d \equiv 1 \ (\pmod 4) \Rightarrow \mathbb{Z}[\sqrt{d}]$ is not a UFD.

9. Let the field S be a subring of R, a ring with 1, and suppose that $1 \in S$ also holds. Prove that R is a vector space of over S, and if R is finite, then $|R| = |S|^n$ for some $n \in \mathbb{N}$.

10. Prove that if for some $I, J \triangleleft R$ ideals $R = I + J$, then $R/(I \cap J) \cong R/I \oplus R/J$.

11. Let $f(x) \in \mathbb{Q}[x]$, and supose that f does not have multiple roots in $\mathbb{C}$. Prove that $\mathbb{Q}[x]/(f(x))$ is a direct sum of fields.

HW1. Let $R = A \oplus B$, $K \triangleleft R$, $1 \in R$. Prove that $K = K \cap A \oplus K \cap B$.

HW2. Let R be an integral domain with 1. Show that $a \in R$ has the prime property if and only if the quotient ring $R/(a)$ has no zero-divisors.

HW3. Find in $\mathbb{Z}[i]$ the irreducible factorization of the number $2 + 6i$.