1. *Consider the permutations* $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 2 & 7 & 1 & 3 & 6 \end{pmatrix}$ *and* $h = (235)(14)$ *in* S_7*. What is the disjoint cycle decomposition of* g*? Calculate* gh *and the conjugate* $h^{-1}gh$*, and then determine the order and parity (whether the permutations are even or odd) of the permutations* g*,* h*,* gh *and* $h^{-1}gh$*.* *(7 points)*

Solution:

element	g	h	gh	$g^h = h^{-1}gh$
dcd	$(15)(24763)$	$(235)(14)$	$(12)(4765)$	$(42)(31765)$
order	10	6	4	10
parity	*odd*	*odd*	*even*	*odd*

2. *Suppose* g *is an element of a non-commutative group* G*, furthermore,* $|G| = 28$*,* $g^7 \neq 1$*, and* $g^8 \neq 1$*. What can be the order of the element* g*?* *(7 points)*

Solution: $o(g) \mid |G| = 28 \Rightarrow o(g) = 1, 2, 4, 7, 14$ or 28. But $g^7 \neq 1 \Rightarrow o(g) \neq 1, 7$ and $g^8 \neq 1 \Rightarrow o(g) \neq 1, 2, 4$, so $o(g)$ is 14 or 28.

3. *Let* G *be the group of invertible* 2×2 *upper triangular matrices over* $\mathbb{Z}_5$*, and* $N = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{Z}_5,\ a \neq 0 \right\} \subseteq G$*. Prove that* N *is a normal subgroup of* G*, and* G/N *is cyclic.* *(7 points)*

Solution:

$$I \in N, \qquad \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & -b/a^2 \\ 0 & 1/a \end{bmatrix} \in N,$$

and

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad + bc \\ 0 & ac \end{bmatrix} \in N,$$

so N is a subgroup. For $g = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in G$ and $n = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$,

$$g^{-1}ng = \begin{bmatrix} 1/x & -y/xz \\ 0 & 1/z \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} a & bz/x \\ 0 & a \end{bmatrix} \in N,$$

so N is a normal subgroup.

$|G| = 4 \cdot 4 \cdot 5 = 80$, $|N| = 4 \cdot 5 = 20 \Rightarrow |G/N| = 80/20 = 4$, so we need to find an element of order 4 to show that G/N is cyclic. For $g = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, the element $Ng \in G/N$ has order 4, since $g, g^2, g^3 \notin N$ but $g^4 = I \in N$.

(A shorter solution makes use of the homomorphism theorem. The map $\varphi : \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mapsto a/c$ is a homomorphism from G to $\mathbb{Z}_5^\times$, since for $g = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $g' = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$

$$gg' = \begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix}, \text{ so } \varphi(gg') = aa'/cc' = (a/c)(a'/c') = \varphi(g)\varphi(g'),$$

and $\operatorname{Ker}\varphi = N$, so N must be a normal subgroup in G. Furthermore, $\mathbb{Z}_5^\times$ is generated by 2 (its powers are $2, 4, 3, 1$), so $G/N = G/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi \leq \mathbb{Z}_5^\times$ gives that G/N is cyclic.

4. *What is the number of elements of order* 12 *in the alternating group* A_{10}*?* *(7 points)*

Solution: In S_{10} an element has order 12 if in its dcd the length of every cycle divides 12 and there is at least one divisible by 4, and at least one divisible by 3. The corresponding partitions can be $4+3$, $4+3+3$, $4+3+2$ and $4+6$. Among these only $4+3+2$ and $4+6$ belong to even permutations. The number of permutations of the form (....)(...)(..) in A_{10} is $\binom{10}{4} \cdot 3! \cdot \binom{6}{3} \cdot 2! \cdot \binom{3}{2}$, those of the form (....)(......) is $\binom{10}{4} \cdot 3! \cdot 5!$. So the number of elements of order 12 in A_{10} is $\binom{10}{4} \cdot 3! \cdot \binom{6}{3} \cdot 2! \cdot \binom{3}{2} + \binom{10}{4} \cdot 3! \cdot 5!$.

5. *a) Prove that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two conjugacy classes of a group G then the set $\mathcal{C}_1 \cdot \mathcal{C}_2 = \{\, gh \mid g \in \mathcal{C}_1,\ h \in \mathcal{C}_2 \,\}$ is the union of some conjugacy classes of G.*

b) For $G = S_4$, the conjugacy class of 2-cycles, that is, $\mathcal{C}_1 = (12)^G$ and the conjugacy class of 3-cycles, that is, $\mathcal{C}_2 = (123)^G$, how many conjugacy classes of G does the set $\mathcal{C}_1 \cdot \mathcal{C}_2$ contain? *(7 points)*

Solution: a) $x^{-1}ghx = (x^{-1}gx)(x^{-1}hx) \in \mathcal{C}_1 \cdot \mathcal{C}_2$ if $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$, so $\mathcal{C}_1 \cdot \mathcal{C}_2$ is closed under conjugation by elements of G.

b) The disjoint cycle decomposition of the product of a 2-cycle and a 3-cycle can be two kinds: if they have one common moved element: $(ab)(acd) = (abcd)$, or they have two common moved elements: $(ab)(abc) = (ac)$ (they cannot be disjoint in S_4, and the order of the elements in the 2-cycle doesn't make a difference). So $\mathcal{C}_1 \cdot \mathcal{C}_2$ is the union of two conjugacy classes: the conjugacy class of the 4-cycles and that of the 2-cycles. (An alternative solution: The product of an odd and an even permutation can only be odd, so $\mathcal{C}_1 \cdot \mathcal{C}_2$ contains only 2- and 4-cycles. This set has at least 8 elements, since $\mathcal{C}_2$ itself has 8 elements, so it cannot be just one of those (6-element) conjugacy classes. Thus, by part a), $\mathcal{C}_1 \cdot \mathcal{C}_2$ is the union of two conjugacy classes.)

6. *Let $H, K \leq G$, $|G| = 80$, $|H| = 20$, $|K| = 8$. Prove that $\langle H \cup K \rangle \lhd G$.* *(7 points)*

Solution: Let $L := \langle H \cup K \rangle$. Then $L \geq H, K$, so $20 \mid |L|$ and $8 \mid |L|$, which gives $\operatorname{lcm}(20, 8) = 40 \mid L$. This implies that $|G : L| = 2$ or 1. But every subgroup of index 2 is normal, and in the second case, $L = G$ is also normal.