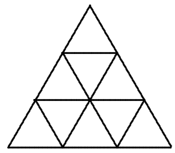
1. *Consider the regular triangle below divided into 9 small congruent triangles. How many ways are there to colour three of the small triangles black, up to isometries of the big triangle?*



(7 points)

Solution: The number of all colourings is $\binom{9}{3} = 84$. The group of isometries D_3 acts on these colourings, and the number of essentially different colourings is the number of orbits of this group action. We use the orbit-counting lemma. The elements of D_3 and their number of fixed-points are:

1:	84
rotations by $\pm 120°$:	3
reflections:	$1 + 3 \cdot 3 = 10$

(For the rotations, if one corner or mid-edge or inside cell is black then all three such cells are black, and nothing else, so there are 3 options. For the reflections, we can either colour the three cells on the axis, or only one on the axis, and chose one on one side and its reflection.)
This gives altogether $\frac{1}{|D_3|}(84 + 2 \cdot 3 + 3 \cdot 10) = \frac{120}{6} = 20$ colourings up to symmetries.

2. *What is the number of elements of order* 4 *in* $D_4 \times Q$*, where* D_4 *is the dihedral group of* 8 *elements and* Q *is the quaternion group?* *(7 points)*

Solution: In a table below are the possible orders of the elements of D_4 and Q, in parantheses the number of elements of that order, and in the intersection the least common multiple of the two orders.

$D_4 \backslash Q$	1 (1)	2 (1)	4 (6)
1 (1)	1	2	4
2 (5)	2	2	4
4 (2)	4	4	4

Then the number of elements of order 4 in $D_4 \times Q$ is $1 \cdot 6 + 5 \cdot 6 + 2 \cdot 8 = 52$.

3. *What is the number of abelian groups of order* 200 *up to isomorphism? List the canonical decompositions of those among them that have elements of order* 4 *but do not have elements of order* 50*?* *(7 points)*

Solution: If G is abelian, and $|G| = 200 = 2^3 \cdot 5^2$ then $G = G_1 \times G_2$, where $|G_1| = 8$ and $|G_2| = 25$, so $G_1 \cong C_8$, $C_4 \times C_2$ or $C_2 \times C_2 \times C_2$, while $G_2 \cong C_{25}$ or $C_5 \times C_5$. This gives $3 \cdot 2 = 6$ possibilities up to isomorphism.
Every element of order 4 must be in G_1, and every cyclic group of order divisible by 4 contains elements of order 4, so G has elements of order $4 \Leftrightarrow G_1 \cong C_8$ or $C_4 \times C_2$ (in $C_2 \times C_2 \times C_2$ the square of every element is 1). $(g_1, g_2) \in G_1 \times G_2$ has order $50 \Leftrightarrow o(g_1) = 2$ and $o(g_2) = 25$, since $|G_1| = 2^3$ and $|G_2| = 5^2$ are coprime. But G_1 always has an element of order 2, so there is no element of order 50 in $G \Leftrightarrow G_2$ has no element of order 25 $\Leftrightarrow G_2 \cong C_5 \times C_5$. The Abelian groups of order 200 satisfying the extra conditions are: $C_8 \times C_5 \times C_5$ and $C_4 \times C_2 \times C_5 \times C_5$.

4. *Let* G *be a group of order* $3^3 \cdot 13$*. What can be the number of Sylow* 3*- and* 13*-subgroups of* G*? Show that one of the Sylow-subgroups is normal.*

Solution: Let $s_3 = |\operatorname{Syl}_3(G)|$, $s_{13} = |Syl_{13}(G)|$, $P_3 \in \operatorname{Syl}_3(G)$ and $P_{13} \in \operatorname{Syl}_{13}(G)$.
$s_3 \mid 13 \Rightarrow s_3 = 1$ or 13, and both satisfiy $s_3 \equiv 1 \pmod 3$.
$s_{13} \mid 27 \Rightarrow s_{13} = 1, 3, 9, 27$, but only $s_{13} = 1$ and 27 of these satisfy $s_{13} \equiv 1 \pmod{13}$.
So $s_3 = 1$ or 13, and $s_{13} = 1$ or 27.
If $s_{13} = 1$ then $P_{13} \lhd G$.
If $s_{13} = 27$, then, since $|P_{13}| = 13$ is a prime, the Sylow 13-subgroups intersect each other trivially, so there are $27 \cdot 12$ elements of order 13 in G. This leaves only $27 \cdot 13 - 27 \cdot 12 = 27$ elements for the Sylow 3-subgroups. But $|P_3| = 27$, so there cannot be any other Sylow 3-subgroups. This implies that in this case $|\operatorname{Syl}_3(G)| = 1$ and $P_3 \lhd G$.

5. *Suppose that the finite group G has a normal subgroup N whose order is a p-power for some prime p. Show that every Sylow p-subgroup of G contains N.* *(7 points)*

Solution: 1^{st} solution: Since N is a p-subgroup, Sylow (1^+) implies that $\exists P \in \operatorname{Syl}_p(G)$: $N \leq P$. Now by Sylow (3), for any $Q \in \operatorname{Syl}_p(G)$ $\exists g \in G$: $Q = P^g$, so $Q = P^g \geq N^g = N$, since $N \lhd G$.
2^{nd} solution: Let $P \in \operatorname{Syl}_p(G)$ any Sylow p-subgroup of G. Then, since $N \lhd G$, $\langle P, N \rangle = PN$, and $|PN| = |P| \cdot |N|/|P \cap N|$ is a p-power. But $P \leq PN$, and $|P|$ is the maximal p-power divisor of $|G|$, so we get that $P = PN$, that is, $N \leq P$.

6. *a) Let I be an ideal in the ring R, and $J = \{\, b \in R \mid ab = 0 \ \forall a \in I \,\}$. Prove that $J \lhd R$.*
b) What is J if R is the group of 3×3 upper triangular matrices over $\mathbb{R}$, and I is the ideal of strictly upper triangular matrices, that is, where all the diagonal elements are zero. *(7 points)*

Solution: a) $0 \in J$ because $a0 = 0$ for every $a \in I$.
If $b \in J$ then $a(-b) = -(ab) = -0 = 0$ for $\forall a \in I$, so $-b \in J$.
If $b, b' \in J$ then $a(b + b') = ab + ab' = 0 + 0 = 0 \ \forall a \in I$, so $b + b' \in J$.
If $b \in J$ and $r \in R$ then $a(br) = (ab)r = 0r = 0 \ \forall a \in I$,
and $a(rb) = (ar)b = 0b = 0 \ \forall a \in I$ since $I \lhd R$ gives $ar \in I$. So $br, rb \in J$.

b)
$$\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} 0 & dx & ex + fy \\ 0 & 0 & fz \\ 0 & 0 & 0 \end{bmatrix}$$

must be the zero matrix for every x, y, z for the second matrix to be in J. This is true $\Leftrightarrow d = e = f = 0$, that is,

$$J = \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$