1. Definitions and examples. (*20 points*)

(a) What is the conjugacy class of an element g of a group G? For $g = (1234) \in S_6$ give an element $h \in S_6$ which is NOT conjugate to g but $o(g) = o(h)$.

(b) Define the Sylow p-subgroup of a group G. What is the size of the Sylow 3-subgroups of S_{10}?

(c) Define the division ring, and give an example for a non-commutative division ring.

(d) Define the ideal of a ring. Give an example of a ring R and a subgring S such that S is not an ideal in R.

2. Theorems. (*20 points*)

(a) State the theorem about the number of elements of given orders in a cyclic group C_n.

(b) State the second isomorphism theorem of groups.

(c) State the orbit-counting lemma for group actions.

(d) State the theorem about prime fields.

(e) What are the possible cardinalities of finite fields? How many such fields exist for a given cardinality?

3. Problems. (*20 points*)

(a) How many elements does the factor ring R/I have if $R = \mathbb{Z}[i] = \{\, a+bi \mid a, b \in \mathbb{Z} \,\} \subseteq \mathbb{C}$ and $I = (2) \triangleleft R$?

(b) Consider the factor ring $L = \mathbb{Q}[x]/(p(x))$, where $p(x) = x^3 - 2$.

(b1) Is L a field?

(b2) Write $\dfrac{1}{\alpha + 1}$ as a polynomial of $\alpha = x + (p(x))$ in L.

4. Proofs. Do ONE of (a) and (b) below. (*18 points*)

(a) Prove the simplicity of A_n (for which n?). Of the four cases in the proof (according to the cycle structure of the chosen element of the normal subgroup), it will be enough to prove one.

(b) State and prove the theorem which describes a simple algebraic extension $K(\alpha)$ as a factor ring.

5. Proofs. Do TWO of (a), (b), (c) below. (*11+11 points*)

(a) State and prove the homomorphism theorem for groups.

(b) State and prove the class equation about center and conjugacy classes.

(c) Let $L|K$ be a field extension. Which of the following two statements implies the other?

(A) $(L : K) < \infty$

(B) $L|K$ is algebraic.

Prove the implication which is true.