Problem Sheet 6

1. Prove that $Z(S_n) = 1$ if $n \geq 3$ and $Z(A_n) = 1$ if $n \geq 4$.

2. Show that if $N \triangleleft G$ and $|N| = 2$ then $N \leq Z(G)$.

3. Prove that the only normal subgroups of S_n are 1, A_n and S_n if $n \geq 5$.

4. Consider the action of the symmetries of the cube on the points of the cube as a solid. What are the sizes of the orbits?

5. How many ways are there to colour the cells of a 3×3 square with black and white up to rotations and reflections (that is, we do not distinguish colorings which can be taken into each other by some isometry of the square), if
a) we want two cells to be black;
b) we want three cells to be black?
What is the answer if we only consider those colourings equivalent which can be rotated into each other?

6. What is the number of simple graphs on five vertices up to isomorphisms? (Use the Orbit-counting lemma for the action of S_5 on the 1024 possible labelled graphs.

HW1. What is the smallest n such that A_n contains an element of order 8? Give an example of such an element, and determine the size of its conjugacy class in A_n.

HW2. How many ways are there to colour four cells of a 4×4 square black, up to symmetries of the square?