Algebra 1 **Problem Sheet 7**

1. What is the centralizer of (123) in A_4, S_4 and S_5.

2. Show that for $g \in A_n$, the conjugacy class of g in S_n is either a conjugacy class in A_n or it is the union of two conjugacy classes of equal size.

3. Prove that $Z(G \times H) = Z(G) \times Z(H)$.

4. Prove that
a) $D_6 \cong D_3 \times C_2$;
b) The group of symmetries of a cube is isomorphic to $S_4 \times C_2$.

5. What is the number of elements of order 4 and 6 in $D_4 \times C_6$?

6. For a group G and integer m let $n_m(G) = |\{ x \in G \,|\, x^m = 1 \}| = |\{ x \in G \,|\, o(x) \mid m \}|$.
a) Let A be a finite abelian p-group with canonical form $A = \prod_{i=1}^{m} A_i$, where A_i are cyclic p-groups, and

$$|A_1| \geq \cdots \geq |A_t| \geq p^k > |A_{t+1}| \geq \cdots \geq |A_m|.$$

Prove that

$$n_{p^k}(A) = |C_{p^k} \times \cdots \times C_{p^k} \times A_{t+1} \times \cdots \times A_m| = (p^k)^t \cdot |A_{t+1}| \cdots |A_m|,$$

and the number of elements of order p^k in A is $n_{p^k}(A) - n_{p^{k-1}}(A)$ for $k \geq 1$.
b) Let $G = P_1 \times \cdots \times P_m$ be the direct product of finite p_i-groups for different primes $p_1, \ldots, p_m$, and $d = p_1^{b_1} \cdots p_m^{b_m}$. Prove that the number of elements of order d in G is the product of the number of elements of order $p_i^{b_i}$ in P_i.

7. a) What is the number of abelian groups of order 32 up to isomorphism? Determine the number of elements of order 4 in each case.
b) What is the number of abelian groups of order 360 up to isomorphism? Determine the number of elements of order 12 in each case.

8. Suppose that $G = \langle a \rangle \times \langle b \rangle$, where $o(a) = 4$, $o(b) = 6$, and $M = \langle a^{-1}b^3 \rangle$, $N = \langle a^2 b^3 \rangle$.
a) What is the order of $\overline{a}$ and $\overline{b}$ in the factor groups G/M and G/N
b) What are the abelian groups G, M, N, G/M and G/N in their canonical form?

HW1. Let $G = GL_2(\mathbb{Z}_3)$ and $g = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \in G$.
a) Determine the centralizer $C_G(g)$, that is, find all invertible matrices $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over $\mathbb{Z}_3$ such that $xg = gx$.
b) What is the size of the conjugacy class of g in G?

HW2. What is the number of Abelian groups of order 600? What is the canonical form of the one among them which has the most elements of order 10?