Algebra 1 **Problem Sheet 9**

1. Prove that G cannot be simple if
a) $|G| = 45$, 56, 80, 36;
b) $|G| = p^a m$, where p is a prime and $p > m > 1$, $a > 0$;
c) $|G| = pq$, $p^2 q$ or $p^2 q^2$, where p, q are prime numbers.

2. Prove that A_5 is the smallest non-abelian simple group.

3. a) What can be the number of Sylow 3-, 5- or 7-subgroups in a group G of order 105?
b) Prove that one of the Sylow subgroups of G must be normal.
c) Prove that the Sylow 7-subgroup is always normal in G.

4. By examining the Sylow subgroups, prove that every group of order 15 is cyclic.

5. Find a group of order 21 in S_7.

6. Consider the pure imaginary elements of the quaternions $\mathbb{H}$ as vectors of $\mathbb{R}^3$ (where i, j, k are the elements of the standard basis). Show that the product of the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ in $\mathbb{H}$ is $-\mathbf{uv} + \mathbf{u} \times \mathbf{v}$, where $\mathbf{uv}$ is the dot product and $\mathbf{u} \times \mathbf{v}$ is the cross product in $\mathbb{R}^3$.

7. Let G be a finite group and $R = KG$ the group algebra of G over K. Prove that $I = K(\sum_{g \in G} g)$ and $J = \{\sum_{g \in G} \lambda_g g \mid \sum_{g \in G} \lambda_g = 0\}$ are both ideals in R, and also subspaces in the vector space KG_K.

8. Prove that $\mathbb{H}$ is not isomorphic to $\mathbb{R}Q$ where Q is the quaternion group.

9. What can we say about a ring R where the set $\{0, a\}$ is an ideal of R for every $a \in R$?

HW1. Let G be a group of order 140. Prove that G has at least two normal Sylow subgroups. Using this, show that G has an element of order 35.

HW2. Prove that the nilpotent elements (that is, the elements r for which there exists a positive integer n with $r^n = 0$) of a commutative ring form an ideal.