Some facts from the arithmetics of polynomials:

Roots and reducibility: *If K is a field, $f(x) \in K[x]$ and $\deg f = 2$ or 3 then f is irreducible $\Leftrightarrow$ f has no root in K.*
It is not true for polynomials of higher degree!!

Rational root test: *If $f(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ ($a_n, a_0 \neq 0$) and $\frac{p}{q} \in \mathbb{Q}$ ($p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$) is a root of f then $p \mid a_0$ and $q \mid a_n$.*

Gauss lemma: *If $f(x) \in \mathbb{Z}[x]$ is reducible in $\mathbb{Q}[x]$ then it can also be factored into a product of polynomials of smaller degree over $\mathbb{Z}[x]$.*

Schönemann–Eisenstein criterion: *Suppose that $f(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$, and there exists a prime p such that p divides $a_{n-1}, \ldots, a_0$ but p does not divide a_n, and p^2 does not divide a_0 then $f(x)$ is irreducible in $\mathbb{Q}[x]$.*

1. Which of the following polynomials are irreducible over $\mathbb{Q}$?
a) $2x - 3$ b) $x^3 - 2x^2 + x + 1$ c) $x^4 + 4x + 3$
d) $x^5 + 2x - 6$ e) $x^4 + 4$ f) $x^4 - x^2 + 1$

2. Find the minimal polynomial of α, that is, an irreducible polynomial of $\mathbb{Q}[x]$ which has α as a root if α is
a) $i\sqrt{3}$ b) $\sqrt[3]{2}$ c) $\sqrt{1 + \sqrt{3}}$ d) $\sqrt[3]{2} + \sqrt[3]{4}$
Do not forget to prove that the polynomial is, indeed, irreducible.

3. Show that $\mathbb{Q}[x]/(z^2 - 2x - 3) \cong \mathbb{Q} \oplus \mathbb{Q}$

4. Let R be the ring of $n \times n$ upper triangular matrices over a field K. Prove that the matrices where all elements which are not in the upper right corner are zero form an ideal J, and every nontrivial ideal of R contains J. Consequently, R cannot be written as a direct sum of nontrivial rings.

5. Prove that $R = \mathbb{R}[x]/((x - 1)^2)$ is isomorphic to the ring $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$.

6. Let $p(x) = a(x)b(x)$ where, $p(x), a(x), b(x) \in K[x]$ for some field K, and assume that $\gcd(a(x), b(x)) = 1$. Prove that

$$K[x]/(p(x)) \cong K[x]/(a(x)) \oplus K[x]/(b(x)).$$

7. Let $\mathbb{Z}[i] = \{ a + bi \mid a, b \in \mathbb{Z} \} \subseteq \mathbb{C}$ be the set of gaussian integers.
a) Prove that $\mathbb{Z}[i]$ is a euclidean ring with norm $N(a + bi) = a^2 + b^2$.
b) Find an element $u \in \mathbb{Z}[i]$ such that the ideal $(1 + 3i, 5) = (u)$.

HW1. Determine the minimal polynomial of $\mathbb{Q}(\sqrt{4 - \sqrt{2}})$ over $\mathbb{Q}$.

HW2. Prove that $\mathbb{R}[x]/(x^3 + x) \cong \mathbb{C} \oplus \mathbb{R}$.