1. *Describe the following factor groups (find a known group or a naturally definable subgroup of such a group isomorphic to the given factor group).*
a) $\mathbb{R}^\times / \langle \pm 1 \rangle$
b) $\langle a \rangle / \langle a^n \rangle$ *if* $G = \langle a \rangle \cong C_\infty$
c) $S_4 / \langle (12)(34), (13)(24) \rangle$

Solution: a) $\mathbb{R}^+ := \{ x \in \mathbb{R} \,|\, x > 0 \}$ is a subgroup of $\mathbb{R}^\times$, $\mathbb{R}^+ \cap \{ \pm 1 \} = \{ 1 \}$, and $\{ \pm 1 \} \mathbb{R}^+ = \mathbb{R}$ is also obvious, so $\mathbb{R}^+$ is a complement of the normal subgroup $\langle \pm 1 \rangle$. This implies theat $\mathbb{R}^\times / \{ \pm 1 \} \cong \mathbb{R}^+$.

b) For $n = 0$, we have $\langle a \rangle / 1 = \langle a \rangle \cong C_\infty$, and $\langle a^n \rangle = \langle a^{-n} \rangle$, so assume that $n > 0$. The (normal) subgroup $N = \langle a^n \rangle = \{ a^{mn} \,|\, m \in \mathbb{Z} \}$ has no complement in $\langle a \rangle \cong C_\infty$ because any subgroup $H \neq 1$ is generated by a^k for some $k > 0$, and then $1 \neq a^{kn} \in N \cap H$. So we try to find a nice and natural transversal and determine the multiplication rule for the corresponding cosets. The elements $1, a, \ldots, a^{n-1}$ are in different cosets for N since for $0 \leq i < j < n$, the element $a^j (a^i)^{-1} = a^{j-i} \notin N$. On the other hand, every coset has an element a^r with $0 \leq r \leq k-1$ because the euclidean division $m = nq + r$ gives $Na^m = N(a^{nq})a^r = Na^r$. So $\{ 1, a, \ldots, a^{n-1} \}$ is a transversal for N. The multiplication is $Na^i Na^j = Na^{i+j} = Na^r$ where r is the remainder of $i + j$ modulo n. This is the same multiplication rule as in C_n, so $G/N \cong C_n$.
(There is a shorter proof which uses the homomorphism theorem. We have seen in problem 1.a) that for $H = \langle b \rangle \cong C_n$ there is a homomorphism $\varphi : G \to H$ with $\varphi(a) = b$. This is clearly surjective, and $a^k \in \operatorname{Ker} \varphi \Leftrightarrow \varphi(a^k) = 1 \Leftrightarrow b^k = \varphi(a)^k = 1 \Leftrightarrow n \mid k$, so $\operatorname{Ker} \varphi = \langle a^n \rangle$, and then by the homomorphism theorem, $G / \langle a^n \rangle = G / \operatorname{Ker} \varphi \cong \operatorname{Im} \varphi = \langle b \rangle \cong C_n$.)

c) We have seen in problem 4/7.b) that $V = \{ 1, (12)(34), (13)(24), (14)(23) \} = \langle (12)(34), (13)(24) \rangle$ is a normal subgroup of S_4. If it has a complement H then $|H| = |S_4|/|V| = 6$. We can easily find a subgroup of 6 elements in S_4, for example $H = \{ g \in S_4 \,|\, 4g = 4 \}$ is such a subgroup, in fact, $H = S_3$. Luckily, $H \cap V = 1$, so it also follows that $|VH| = |V| \cdot |H| / |V \cap H| = 24$, implying that $S_4 = VH$, thus H is a complement of V, consequently, $S_4/V \cong H \cong S_3$.

2. *Determine the factor groups of* D_4.

Solution: $|D_4| = 8$, so the order of every subgroup of D_4 is 1, 2, 4 or 8.
If $|H| = 1$ then $H = 1$, and $D_4/1 = D_4$.
If $|H| = 8$ then $H = D_4$, and $D_4/D_4 \cong 1$.
If $|H| = 4$ then $|D_4 : H| = 2$, so H must be a normal subgroup, and $|D_4/H| = 2$. But every group of prime order is cyclic, thus $D_4/H \cong C_2$.
Finally, suppose that $|H| = 2$. If it is a normal subgroup then besides $\{ 1 \}$, there must be another 1-element conjugacy class in it. The only such conjugacy class is $\{ r^2 \}$ (by problem 4/8.a)), thus $H = \langle r^2 \rangle = Z(D_4)$. Clearly, $|D_4 / \langle r^2 \rangle| = 4$, and every nontrivial element of the factor group has order 2: the square of the reflections is 1, and $(r^{-1})^2 = r^2 \in \langle r^2 \rangle$. This is the same situation as in the Klein group $V = \{ 1, (12)(34), (13)(24), (14)(23) \}$ It also follows that the product of any two different non-identity elements of $D_4 / \langle r^2 \rangle$ is the third such element because it cannot be equal to any of the two or the identity. So any bijection

$D_4/\langle r^2\rangle \to V$ which takes 1 to 1 is an isomorphis between the two groups, proving that $D_4/\langle r^2\rangle \cong V$.

3. *Let G be a group and $H \le G$. Are the following statements true or false?*

a) There always exists a group L and homomorphism $\varphi : G \to L$ such that $\operatorname{Ker}\varphi = H$.
b) If $H \lhd G$, then there exists a group L and homomorphism $\varphi : G \to L$ such that $\operatorname{Ker}\varphi = H$.
c) There always exists a group L and homomorphism $\varphi : L \to G$ such that $\operatorname{Im}\varphi = H$.
d) If $\varphi : G \to L$ is a homomorphism then $\varphi(H) \le L$.
e) If $H \lhd G$, and $\varphi : G \to L$ is a homomorphism then $\varphi(H) \lhd L$.
f) If $H \lhd G$, and $\varphi : G \to L$ is a homomorphism then $\varphi(H) \lhd \varphi(G)$.
g) If $\varphi : G \to L$ is a homomorphism then $|G| \mid |\varphi(G)|$.
h) If $\varphi : G \to L$ is a homomorphism then $|\varphi(G)| \mid |G|$.
i) A group L is a homomorphic image of G (that is, there exists a surjective homomorphism $G \to L$) if and only if L is isomorphic to a factor group of G.

Solution: a) False. The kernel must be a normal subgroup.
b) True, $\varphi : G \to G/H$, $\varphi(g) = Hg$ is such a homomorphism.
c) True, $\varphi : H \to G$, $\varphi(h) = h$ is such a homomorphism.
d) True because $1 = \varphi(1) \in \varphi(H)$, and for $h, h' \in H$ we have $\varphi(h)\varphi(h')^{-1} = \varphi(h(h')^{-1}) \in \varphi(H)$.
e) False. If $S \le L$ is not a normal subgroup then for the homomorphism $\varphi : S \to L$, $\varphi(s) = s$ it is true that $S \lhd S$ but $\varphi(S) = S$ is not a normal subgroup in L.
f) True. We have seen in part d) that $\varphi(H)$ is a subgroup in L (and it is clearly contained in $\varphi(G)$), furthermore, for any $\varphi(g) \in \varphi(G)$ and $\varphi(h) \in \varphi(H)$ (where $h \in H$) $\varphi(g)^{-1}\varphi(h)\varphi(g) = \varphi(g^{-1}hg) \in \varphi(H)$ because $g^{-1}hg \in H$.
g) False, for example, if $|G| \ne 1$, and $\varphi = 1$ then $|\varphi(G)| = 1$ is not divisible by $|G|$.
h) True because $\varphi(G) = \operatorname{Im}\varphi \cong G/\operatorname{Ker}\varphi$, so $|\varphi(G)| = \frac{|G|}{|\operatorname{Ker}\varphi|} \mid |G|$.
i) True. The homomorphism theorem ensures that if $\varphi : G \to L$ is surjective then $G/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi = L$. Conversely, if $\varphi : G/N \to L$ is an isomorphism for some $N \lhd G$ then with the natural homomorphism $\psi : G \to G/N$, $g \mapsto Ng$, the image of $\varphi \circ \psi$ is L.

4. *Let $H \le G$ and $M, N \lhd G$. Prove that*

a) $H \cap N \lhd H$; $N \cap M \lhd G$;
b) $HN \le G$; $NM \lhd G$;
c) if $M \le N$ then G/N is a homomorphic image of G/M.

Solution: a) The intersection of subgroups is a subgroup so we only have to prove that they are closed under conjugation.
$x \in H \cap N$, $h \in H \Rightarrow h^{-1}xh \in H$ because $h, x \in H$, and $h^{-1}xh \in N$ because $x \in N \lhd G$, so $h^{-1}xh \in H \cap N$.
For $x \in N \cap M$, $g \in G$, $g^{-1}xg \in N$ and $\in M$, since N and M are both normal, so $g^{-1}xg \in N \cap M$.
b) $N \lhd G \Rightarrow HN = NH \Rightarrow HN \le G$.
From this it also follows that $NM \le G$. Furthermore, NM is closed under conjugation by elements of G, since for $g \in G$, $n \in N$, $m \in M$ we have $g^{-1}nmg = (g^{-1}ng)(g^{-1}mg) \in NM$.

c) By the 2^{nd} isomorphism theorem, $(G/M)/(N/M) \cong G/N$, so G/N is a factor group of G/M, and then by problem 1.i), it is also a homomorphic image of G/M.

5. *Let $N \lhd G$. Prove that the map $H \mapsto H/N := \{ Nh \mid h \in H \}$ is a bijection between the subgroups of G containing N and the subgroups of G/N. Show that this map connects normal subgroups with normal subgroups, and it preserves the inclusion of subgroups, that is, $H_1 \leq H_2$ if and only if the subgroup of G/N corresponding to H_1 is contained in the one corresponding to H_2.*

Solution: Let $S(G)$ and $S(G/N)$ be the set of subgroups of G and G/N, respectively, and $S(G)_{\geq N}$ the set of those subgroups of G which contain N. Let

$$\Phi : S(G) \to S(G/N), \quad \Phi(H) = \{ Nh \mid h \in H \}$$

($\Phi(H)$ is indeed a subgroup, since $N1 = N$ is the identity element of G/N, $Nh_1Nh_2 = Nh_1h_2$, and $(Nh)^{-1} = Nh^{-1}$). Conversely,

$$\Psi : S(G/N) \to S(G), \quad \Psi(\{ Ng_i \mid i \in I \}) = \bigcup_{i \in I} Ng_i$$

(this also forms a subgroup, because $1 \in N$, and for $a \in Ng_i$ and $b \in Ng_j$, we have $ab \in Ng_iNg_j$ and $a^{-1} \in (Ng_i)^{-1}$, where $\bar{1} = N$, Ng_iNg_j and $(Ng_i)^{-1}$ are in the given subgroup of G/N).

Note that the elements of $\operatorname{Im}\Psi$ are subgroups containing N since $\bar{1} = N$ is in every subgroup of G/N.

We show that the restriction of Φ to $S(G)_{\geq N}$ (denote it by Φ_1) and Ψ are inverses of each other.

If $N \leq H \leq G$ then $\Psi(\Phi_1(H)) = \bigcup \{ Nh \mid h \in H \} \subseteq H$, and every element of H is in the union, since $h = 1h \in Nh$, so $\Psi(\Phi_1(H)) = H$.

Conversely, if $\mathcal{H} = \{ Ng_i \mid i \in I \} \in S(G/N)$, then $\Phi_1(\Psi(\mathcal{H})) = \{ Nx \mid x \in Ng_i \text{ for some } i \in I \}$. But for these elements x we have $Nx = Ng_i$, so $\Phi_1(\Psi(\mathcal{H})) = \mathcal{H}$.

We have shown that Φ_1 (and the same way Ψ) is a bijection between $S(G)_{\geq N}$ and $S(G/N)$.

It is clear that Φ_1 (and Φ, too) and Ψ preserves inclusion, that is, if $H_1 \leq H_2$, then $\Phi(H_1) \leq \Phi(H_2)$, and if $\mathcal{H}_1 \leq \mathcal{H}_2$, then $\Psi(\mathcal{H}_1) \leq \Psi(\mathcal{H}_2)$.

If $H \lhd G$, then $\Phi(H)$ is also closed under conjugation: $(Ng)^{-1}(Nh)(Ng) = Ng^{-1}hg \in \Phi(H)$ because $g^{-1}hg \in H$ for every $g \in G$. Conversely, if $\mathcal{H} \lhd G/N$, then for every $g \in G$ and $a \in Ng_i \in \mathcal{H}$, we have $g^{-1}ag \in (Ng)^{-1}Ng_iNg \in \mathcal{H}$.

6. *Prove that if $N \lhd G$, and $|G : N|$ is even then there is an H with $N \leq H \leq G$ such that $|H : N| = 2$.*

Solution: $|G/N|$ is even, so it has an element of order 2 (see problem 3/1, or the Cauchy theorem, which we are going to prove later on in the lecture), and this generates a subgroup $\mathcal{H}$ of order 2. The map Ψ of the previous problem assigns a subgroup H to $\mathcal{H}$ which is the union of two cosets of N ($N \cup Nt$ for $t \in G \setminus N$), so $N \leq H$, and $|H : N| = 2$.

7. *Let $N \lhd G$, $H \leq G$, $|G| = 24$, $|N| = 4$, and $|H| = 6$. What can be the cardinality of H at the natural homomorphism $G \to G/N$? Give an example for each case.*

Solution: The image of H by the natural map $G \to G/N$ is $NH/N \cong H/(N \cap H)$, so its cardinality is $|H|/|N \cap H|$. Since $|N \cap H|$ is a divisor of $|N| = 4$ and of $|H| = 6$, so it divides $\gcd(4, 6) = 2$, hence $|N \cap H| = 1$ or 2. Thus the cardinality of the image of H is either 6 or 3.
Both cases are possible. If $G = S_4$, $N = \langle 1, (..)(..) \rangle$ is the Klein group and $H = S_3 = \langle (12)(123) \rangle$ then $N \cap H = 1$, so the image of H has cardinality 6. If $G = \langle a \rangle \cong C_{24}$, $N = \langle a^6 \rangle$ and $H = \langle a^4 \rangle$ then $N \cap H = \langle a^{12} \rangle \cong C_2$, so the image of H has cardinality 3.

8. *What is the number of elements of order* 4 *in* A_8*?*

Solution: In the disjoint cycle decomposition of an element g of order 4, there is at least one 4-cycle, and apart from the 4-cycles, there are only 2-cycles and fixed-points. If $g \in A_n$ then g has evenly many cycles of even length, so it either contains two 4-cylces or one 4-cycle and one 2-cycle. Of the first type, there are $\binom{8}{4}(3!)^2 \cdot \frac{1}{2} = 1260$ elements in A_8, of the second, there are $\binom{8}{4}3!\binom{4}{2} = 2520$ elements, so A_8 contains altogether 3780 elements of order 4.

9. *Prove that the elements* (123) *and* (234) *of* A_4 *are conjugate in* S_4 *but they are not conjugate in* A_4.

Solution: Let $g = (123) = (123)(4)$ and $h = (234) = (234)(1)$. The two elements have the same cycle structure, so they are conjugate in S_4.
The element h can be written in three different ways, keeping the order of the cycle lengthes aligned with those of g: we can permute the elements in the 3-cycle cyclically. We can determine the conjugating elements of S_4 according to the given form of h.

$$\begin{array}{ccc} (123)(4) & (123)(4) & (123)(4) \\ (234)(1) & (342)(1) & (423)(1) \end{array}$$

The corresponding conjugating elements (that map every element of the base set to the one below it) are (1234), (1324), and (14), respectively, and neither of them are in A_4, so g and h are not conjugate in A_4.

HW1. *Let* $G = \langle a \rangle \cong C_{24}$ *and* $N = \langle a^{40} \rangle$*. Determine the order of the (normal) subgroup* N *and of the factor group* G/N*. Find an element* $g \in G$ *such that* $o(\overline{g}) = 4$ *in* G/N *but* $o(g) \neq 4$ *in* g.

HW2. *Suppose that* $N \lhd G$, $H \leq G$, $|G| = 100$, $|N| = 20$ *and* $|H| > 20$*. Prove that there is a subgroup* $L \leq H$ *with* $|H : L| = 5$*. (Hint: what can be the order of* NH*?)*