1. *Prove that $Z(S_n) = 1$ if $n \geq 3$ and $Z(A_n) = 1$ if $n \geq 4$.*

Solution: If $1 \neq g \in S_n$ then there are distinct elements α, β in the base set Ω such that $\alpha g = \beta$, and $n \geq 3$ implies that there also exists a third element in Ω, let it be γ. Then for the transposition $h = (\beta\gamma)$, $g^h : \alpha \mapsto \gamma$, so $g \neq g^h$, consequently, $g \notin Z(S_n)$. Thus $Z(G) = 1$.

Now let us assume that $n \geq 4$, and $1 \neq g \in A_n$. Then again there are distinct elements α, β in the base set such that $\alpha g = \beta$, and there are at least two more elements, say, γ and δ in the base set. The 3-cycle $h = (\beta\gamma\delta)$ is in A_n, and $g^h : \alpha \mapsto \gamma$, so $g^h \neq g$, thus $g \notin Z(A_n)$. It follows that $Z(A_n) = 1$.

(Note that the lower bound for n was necessary because $S_2 \cong C_2$ and $A_3 \cong C_3$, so there the center is not trivial.)

2. *Show that if $N \triangleleft G$ and $|N| = 2$ then $N \leq Z(G)$.*

Solution: Let $N = \{1, t\}$. For any $g \in G$, we have $1 \neq t^g \in N$, so $t^g = t$, proving that $t \in Z(G)$, consequently, $N \leq Z(G)$.

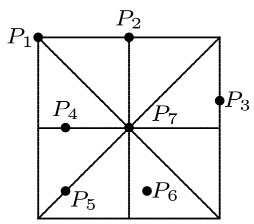
3. *Prove that the only normal subgroups of S_n are 1, A_n and S_n if $n \geq 5$.*

Solution: Suppose $1 \neq N \triangleleft S_n$. Then $A_n \cap N \triangleleft A_n$. However, A_n is simple, so either $A_n \cap N = A_n$ or $A_n \cap N = 1$. In the first case, $A_n \leq N \leq S_n$, and $|S_n : A_n| = 2$ is a prime, so by the Lagrange theorem N can only be A_n or S_n. In the second case, $A_n < A_nN \leq S_n$ implies that $A_nN = S_n$, so $2 = |A_nN/A_n| = |N/N \cap A_n| = |N|$. By problem 2, $N \leq Z(S_n)$, but by problem 1, $Z(S_n) = 1$, so there is a contradiction in this case.

4. *Consider the action of the symmetries of the cube on the points of the cube as a solid. What are the sizes of the orbits?*

Solution: We have proved in problem 1/1.c) that the order of the group of symmetries G of the cube is 48, so the size of every orbit must be a divisor of 48 (of course, there are infinitely many orbits). Note that any point inside the cube, which is different from the center, is on the surface of a concentric cube, which is also invariant to the symmetries of the big cube, thus the orbit sizes are the same for those as for the points on the surface of the big cube. Thus we only have to examine the center and the surface points.

The center is kept fixed by any symmetry, so it forms a 1-element orbit in itself. A surface point is on one of the faces.



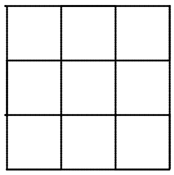
The picture shows the different types of positions of a surface point.

We know that every face can be mapped to any face, reflected or rotated, and any edge can be mapped to any edge in both directions, using symmetries of the cube. Using this, we can count the number of the possible images of the points $P_1, \ldots, P_7$, that is, the sizes of their orbits. Let us denote the orbit of P_i by $\mathcal{O}(P_i)$.

$|\mathcal{O}(P_1)| = 8$, $|\mathcal{O}(P_2)| = 12$, $|\mathcal{O}(P_3)| = 24$, $|\mathcal{O}(P_4)| = 24$, $|\mathcal{O}(P_5)| = 24$, $|\mathcal{O}(P_6)| = 48$, $|\mathcal{O}(P_7)| = 6$.
So the possible orbit sizes are 1, 6, 8, 12, 24, 48. Of the divisors of 48 only 2, 3, 4 and 16 are missing.

5. *How many ways are there to colour the cells of a 3×3 square with black and white up to rotations and reflections (that is, we do not distinguish colourings which can be taken into each other by some isometry of the square), if*
a) we want two cells to be black;
b) we want three cells to be black?
What is the answer if we only consider those colourings equivalent which can be rotated into each other?

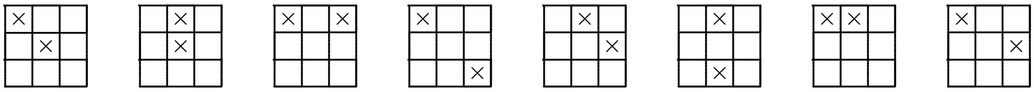
Solution: It is easier to determine the number of colourings if we do not consider the rotated and reflected versions to be the same: in a) it is $\binom{9}{2} = 36$, in b), it is $\binom{9}{3} = 84$. Clearly, D_4 maps a good colouring to another good colouring, so we get a group action on the base sets of 36 and 84 colourings, respectively. Then we want to put together those which can be mapped to each other by D_4, that is, the elements of the orbits by the group action. So the number of essentially different colourings is the number of orbits. By the orbit-counting lemma, the number of orbits is the average of the number of fixed-points of the elements of D_4 at the given group action.



a) For each element of D_4 we have to count the colourings that stay fixed when we apply the given isometry. Clearly, 1 keeps all 36 colourings fixed. If r is the rotation by $90°$ (or $-90°$), then r cannot have any fixed point because in the fixed colouring there must be at least one non-central black cell, and then all others obtained by iterations of r must also be black and that would mean at least 4 black cells. For r^2 with any black cell, its central reflection must also be black, so for a noncentral cell it means already two, so the central cell cannot be black. This means that we have to choose one black cell among the ones in positions $(1,1), (1,2), (1,3), (2,1)$, and then we add its image by r^2. This gives 4 possibilities. Similarly, the number of fixed colourings for a reflection across a bisector of an edge of the square is $3 + \binom{3}{2} = 6$ (we either choose a black cell on one side and its image, or we choose two on the axis of symmetry), and we get the same for the reflections across diagonals because the diagonal also goes through 3 cells, and there are 3 and 3 cells on both sides. Thus the number of colourings up to rotations and reflections is:

$$\frac{1}{8}(36 + 2 \cdot 0 + 4 + 2 \cdot 6 + 2 \cdot 6) = \frac{64}{8} = 8.$$

Actually, these 8 cases are easy to list:



The sizes of their orbits are 4, 4, 4, 2, 4, 2, 8 and 8 (altogether 36, the total number of colourings). If we only consider those equivalent which can be moved to each other by rotation then the number of colourings is

$$\frac{1}{4}(36 + 2 \cdot 0 + 4) = 10$$

(the last two of the eight have reflected versions).

b) Here the number of fixed colourings are:

$$\begin{array}{lr} 1: & \binom{9}{3} = 84 \\ r, r^{-1}: & 0 \\ r^2: & 4 \\ \text{refl. across bisectors}: & 3 \cdot 3 + 1 = 10 \\ \text{refl. across diagonals}: & 3 \cdot 3 + 1 = 10 \end{array}$$

So the number of colourings up to reflections and rotations is

$$\frac{1}{8}(84 + 2 \cdot 0 + 4 + 2 \cdot 10 + 2 \cdot 10) = \frac{128}{8} = 16,$$

and the number of colourings up to rotations is

$$\frac{1}{4}(84 + 2 \cdot 0 + 4) = \frac{88}{4} = 22.$$

6. *What is the number of simple graphs on five vertices up to isomorphisms? (Use the Orbit-counting lemma for the action of S_5 on the 1024 possible labelled graphs.*

Solution: Since there are $\binom{5}{2} = 10$ unordered pairs of vertices in a graph on 5 points, there are $2^{10} = 1024$ ways to decide which pair will be an edge and which pair will not. Two of these graphs are isomorphic if and only if there is a permutation of the vertices that takes one to the other. So we have to consider the action of $G = S_5$ on the 1024 graphs and count its orbits.

Clearly, conjugate permutations have the same number of fixed graphs: we get one from the other by relabelling the vertices. So when we use the Orbit-counting lemma, we only have to count the fixed-points for one permutation in each conjugacy class, that is, one for each cycle structure. The following are the conjugacy classes, their sizes and the number of fixed graphs for a representative of the conjugacy class.

g^G	1	$(12345)^G$	$(1234)^G$	$(123)(45)^G$	$(123)^G$	$(12)(34)^G$	$(12)^G$
$\lvert g^G\rvert$	1	24	30	20	20	15	10
$\lvert \mathrm{Fix}(g)\rvert$	2^{10}	2^2	2^3	2^3	2^4	2^6	2^7

Details for $|\operatorname{Fix}(g)|$:
A graph is a fixed graph of g if and only if g maps edges to edges and non-edges to non-edges. That is, if we consider the action of $\langle g\rangle$ on the unordered pairs of vertices, the orbits should be uniformly chosen to be edges or non-edges. This means, that $|\operatorname{Fix}(g)| = 2^k$ if k is the number of orbits of the action of $\langle g\rangle$ on the 10 unordered pairs of vertices (for simplicity, we denote them by ij or ji for the vertices i and j). For the given permutations g these orbits are:
1: all 1-element sets.
(12345): $\{12, 23, 34, 45, 51\}$, $\{13, 24, 35, 41, 52\}$
(1234): $\{12, 23, 34, 41\}$, $\{13, 24\}$, $\{15, 25, 35, 45\}$
(123)(45): $\langle(123)(45)\rangle = \langle(123), (45)\rangle$, so the orbits are
$\{12, 23, 31\}$, $\{45\}$, $\{14, 24, 34, 15, 25, 35\}$
(123): $\{12, 23, 31\}$, $\{14, 24, 34\}$, $\{15, 25, 35\}$, $\{45\}$.
(12)(34): $\{12\}$, $\{34\}$, $\{13, 24\}$, $\{14, 23\}$, $\{15, 25\}$, $\{35, 45\}$.
(12): $\{13, 23\}$, $\{14, 24\}$, $\{15, 25\}$, $\{12\}$, $\{34\}$, $\{35\}$, $\{45\}$.

So the number of graphs up to isomorphisms is

$$\frac{1}{120}(1\cdot 1024 + 24\cdot 4 + 30\cdot 8 + 20\cdot 8 + 20\cdot 16 + 15\cdot 64 + 10\cdot 128) = 34.$$

HW1. *What is the smallest n such that A_n contains an element of order 8? Give an example of such an element, and determine the size of its conjugacy class in A_n.*

HW2. *How many ways are there to colour four cells of a 4×4 square black, up to symmetries of the square?*