

1. What is the centralizer of (123) in A_4 , S_4 and S_5 .

Solution: Let $x = (123)$. We want to find the permutations $g \in A_4$, S_4 or S_5 , for which $x^g = x$. We have as many different conjugating elements in the symmetric groups as the number of ways to write x in S_4 or S_5 , keeping the cycle structure $3 + 1$ and $3 + 1 + 1$, respectively. In the first case, only the 3-cycle can be rotated, in the second, the two fixed-points can also be swapped, so $|C_{S_4}(x)| = 3$ and $|C_{S_5}(x)| = 6$. The element x clearly centralizes itself, so $\langle x \rangle \cong C_3$ must be in the centralizer. It follows from the sizes that then $C_{S_4}(x) = \langle (123) \rangle$, and in $C_{S_5}(x)$ we still have to find an element of order 2. Clearly, (45) is there, so $C_{S_5}(x) = \langle (123), (45) \rangle$. Finally, $C_{A_4}(x) = A_4 \cap C_{S_4}(x) = \langle (123) \rangle$.

Alternatively, we can find the elements of the centralizer by determining the conjugating permutation for each form of x .

$$\begin{array}{ccc} (123)(4) & (123)(4) & (123)(4) \\ (123)(4) & (231)(4) & (312)(4) \end{array}$$

gives $C_{S_4}(x) = \{1, (123), (132)\} = C_{A_4}(x)$ and

$$\begin{array}{cccccc} (123)(4)(5) & (123)(4)(5) & (123)(4)(5) & (123)(4)(5) & (123)(4)(5) & (123)(4)(5) \\ (123)(4)(5) & (231)(4)(5) & (312)(4)(5) & (123)(5)(4) & (231)(5)(4) & (312)(5)(4) \end{array}$$

gives $C_{S_5}(x) = \{1, (123), (132), (45), (123)(45), (132)(45)\}$.

2. Show that for $g \in A_n$, the conjugacy class of g in S_n is either a conjugacy class in A_n or it is the union of two conjugacy classes of equal size.

Solution: Since A_n is a normal subgroup, it is the union of whole conjugacy classes of S_n but some elements that can be conjugated into each other by a permutation in S_n may not be conjugated into each other by an element in A_n (see problem 5/9 for a counterexample). Let $H = C_{S_n}(g)$. Then $C_{A_n}(g) = A_n \cap H$. Clearly, $A_n \leq A_n H \leq S_n$, and $|S_n : A_n| = 2$, so either $A_n H = A_n$ or $A_n H = S_n$. It follows from the formula $|A_n H| = |A_n||H|/|A_n \cap H|$ that $|g^{A_n}| = |A_n : C_{A_n}(g)| = |A_n|/|A_n \cap H| = |A_n H|/|H| = |A_n H : C_{S_n}(g)|$ is either $|S_n : C_{S_n}(g)| = |g^{S_n}|$ or half of it. This means that the conjugacy classes of A_n that fall into one conjugacy class of S_n are either the whole conjugacy class or two conjugacy classes of equal size.

3. Prove that $Z(G \times H) = Z(G) \times Z(H)$.

Solution:

$$(u, v) \in Z(G \times H) \Leftrightarrow (u, v)(g, h) = (g, h)(u, v) \quad \forall g \in G, h \in H$$

$$\Leftrightarrow (ug, vh) = (gu, hv) \quad \forall g \in G, h \in H$$

$$\Leftrightarrow ug = gu \quad \forall g \in G \text{ and } vh = hv \quad \forall h \in H$$

$$\Leftrightarrow u \in Z(G) \text{ and } v \in Z(H) \Leftrightarrow (u, v) \in Z(G) \times Z(H).$$

4. Prove that

a) $D_6 \cong D_3 \times C_2$;

b) The group of symmetries of a cube is isomorphic to $S_4 \times C_2$.

Solution: a) Let the vertices of the regular hexagon be $1, 2, 3, 4, 5, 6$ (the edges being $12, 23, \dots, 61$), and $H = \{g \in D_6 \mid g \text{ leaves the triangle } 135 \text{ fixed}\}$. Since any isometry of the triangle is also an isometry of the hexagon, $S_3 \cong H \leq D_6$. Then $|D_6 : H| = \frac{12}{6} = 2$, so $H \triangleleft D_6$. On the other hand, the rotation by 180° , denoted by r^3 is in $Z(D_6)$, so $C_2 \cong \langle r^3 \rangle \triangleleft D_6$. Furthermore, $\langle r^3 \rangle \cap H = 1$, so by the orders, $H \langle r^3 \rangle = H \times \langle r^3 \rangle = D_6$, thus $D_6 \cong S_3 \times C_2$.

b) Let G be the group of isometries of the cube. By problem 1/1.c), $|G| = 48$. The endpoints of the diagonals of the faces adjacent to one vertex of the cube span a regular tetrahedron (each of the six edges is the diagonal of a face of the cube, so they have the same length). Let the set of isometries of this tetrahedron be H . These isometries map the cube to itself, so $H \leq G$. Since the group action of H on the vertices of the tetrahedron is a bijection to S_4 , we have $H \cong S_4$. Similarly to part a), we get that $H \triangleleft G$. The central reflection t of the cube is in $Z(G)$ (the elements of G are linear transformations of the vector space with its origin in the center of the cube, and the matrix of t is $-I$, which clearly commutes with all matrices in $GL_3(\mathbb{R})$), giving $C_2 \cong \langle t \rangle \triangleleft G$. Furthermore, $H \cap \langle t \rangle = 1$, hence $G = H \times \langle t \rangle \cong S_4 \times C_2$.

5. What is the number of elements of order 4 and 6 in $D_4 \times C_6$?

Solution: We know that the order of $(g, h) \in G \times H$ is $\text{lcm}(o(g), o(h))$. Let's write in a table the possible orders of the elements in the components D_4 and C_6 , in parantheses the number of elements of that order, and in the intersection the least common multiple of the two orders.

$D_4 \backslash C_6$	1 (1)	2 (1)	3 (2)	6 (2)
1 (1)	1	2	3	6
2 (5)	2	2	6	6
4 (2)	4	4	12	12

Then we can see that the number of elements of order 4 in $D_4 \times C_6$ is $2 \cdot 1 + 2 \cdot 1 = 4$, and the number of elements of order 6 is $1 \cdot 2 + 5 \cdot 2 + 5 \cdot 2 = 22$.

6. For a group G and integer m let $n_m(G) = |\{x \in G \mid x^m = 1\}| = |\{x \in G \mid o(x) \mid m\}|$.

a) Let A be a finite abelian p -group with canonical form $A = \prod_{i=1}^r A_i$, where A_i are cyclic p -groups, and

$$|A_1| \geq \dots \geq |A_t| \geq p^k > |A_{t+1}| \geq \dots \geq |A_r|.$$

Prove that

$$n_{p^k}(A) = |C_{p^k} \times \dots \times C_{p^k} \times A_{t+1} \times \dots \times A_r| = (p^k)^t \cdot |A_{t+1}| \dots |A_r|,$$

and the number of elements of order p^k in A is $n_{p^k}(A) - n_{p^{k-1}}(A)$ for $k \geq 1$.

b) Let $G = P_1 \times \dots \times P_r$ be the direct product of finite p_i -groups for different primes $p_1, \dots, p_r$, and $d = p_1^{b_1} \dots p_r^{b_r}$. Prove that the number of elements of order d in G is the product of the number of elements of order $p_i^{b_i}$ in P_i .

Solution: a) For $g = (g_1, \dots, g_r)$ ($g_i \in A_i$), $g^{p^k} = 1 \Leftrightarrow g_i^{p^k} = 1 \forall i$. Since $|A_i|$ is a p -power smaller than p^k for $i > t$, $o(g_i) \mid |A_i| \mid p^k$, which gives $g_i^{p^k} = 1$ for every $i > t$. On the other hand, for $i \leq t$ the cyclic group A_i contains exactly one subgroup of order p^d

for $d \mid k$, so every element of order dividing p^k is in the unique subgroup $B_i \cong C_{p^k}$ of A_i . Thus $g^{p^k} = 1 \Leftrightarrow g \in B_1 \times \cdots \times B_t \times A_{t+1} \times A_r$, and the order of this subgroup is $(p^k)^t \cdot |A_{t+1}| \cdots |A_r|$.

Finally, since the order of an element $g \in G$ with $g^{p^k} = 1$ is a divisor of p^k , and all proper divisors of p^k divide p^{k-1} , we get that $o(g) = p^k \Leftrightarrow g^{p^k} = 1$ but $g^{p^{k-1}} \neq 1$, which gives that $|\{g \mid o(g) = p^k\}| = n_{p^k}(G) - n_{p^{k-1}}(G)$.

- b) Let $g = (g_1, \dots, g_r)$, where $g_i \in P_i$. Then $o(g_i)$ is a p_i -power, so $o(g) = \text{lcm}(o(g_1), \dots, o(g_r)) = o(g_1) \cdots o(g_r)$. It follows from the unique factorization of natural numbers $o(g) = d \Leftrightarrow o(g_i) = p_i^{b_i}$ for every i .

7. a) What is the number of Abelian groups of order 32 up to isomorphism? Determine the number of elements of order 4 in each case.
 b) What is the number of Abelian groups of order 360 up to isomorphism? Determine the number of elements of order 12 in each case.

Solution: a) The possible decompositions into a direct product of cyclic groups of prime-power orders are

$$C_{32}, \quad C_{16} \times C_2, \quad C_8 \times C_4, \quad C_8 \times C_2 \times C_2, \quad C_4 \times C_4 \times C_2,$$

$$C_4 \times C_2 \times C_2 \times C_2, \quad C_2 \times C_2 \times C_2 \times C_2 \times C_2.$$

So there are 7 Abelian groups of order 32 up to isomorphism.

By 6.a), the number of elements of order 4 is $n_4(G) - n_2(G)$, and we have the following table.

	C_{32}	$C_{16} \times C_2$	$C_8 \times C_4$	$C_8 \times C_2 \times C_2$	$C_4 \times C_4 \times C_2$	$C_4 \times C_2^3$	C_2^5
n_4	4	$4 \cdot 2 = 8$	$4 \cdot 4 = 16$	$4 \cdot 2 \cdot 2 = 16$	$4 \cdot 4 \cdot 2 = 32$	$4 \cdot 2^3 = 32$	$2^5 = 32$
n_2	2	$2 \cdot 2 = 4$	$2 \cdot 2 = 4$	$2 \cdot 2 \cdot 2 = 8$	$2 \cdot 2 \cdot 2 = 8$	$2 \cdot 2^3 = 16$	$2^5 = 32$
$o(g)=4$	2	4	12	8	24	16	0

- b) $|G| = 360 = 2^3 \cdot 3^2 \cdot 5$.

$G = G_1 \times G_2 \times G_3$ as the product of a 2-group, a 3-group and a 5-group.

$$G_1 \cong C_8 \text{ or } C_4 \times C_2 \text{ or } C_2 \times C_2 \times C_2$$

$$G_2 \cong C_9 \text{ or } C_3 \times C_3$$

$$G_3 \cong C_5$$

So there are $3 \cdot 2 \cdot 1 = 6$ such Abelian groups up to isomorphism.

By problem 6.b), the order of $g = (g_1, g_2, g_3)$ is $12 \Leftrightarrow o(g_1) = 4, o(g_2) = 3$ and $o(g_3) = 1$. In G_1 there are $4 - 2 = 2$ or $8 - 4 = 4$ or 0 such elements, in G_2 there are $3 - 1 = 2$ or $9 - 1 = 8$ such elements, and g_3 can only be 1. So the number of elements of order 12 in the six cases are 4, 8, 0 and 16, 32, 0.

8. Suppose that $G = \langle a \rangle \times \langle b \rangle$, where $o(a) = 4, o(b) = 6$, and $M = \langle a^3 b^3 \rangle, N = \langle a^2 b^3 \rangle$.

a) What is the order of $\bar{a}$ and $\bar{b}$ in the factor groups G/M and G/N

b) What are the abelian groups $G, M, N, G/M$ and G/N in their canonical form?

Solution: a) Both M and N are cyclic. For M the powers of the generator a^3b^3 are a^3b^3 , $(a^3b^3)^2 = a^6b^6 = a^2$, $(a^3b^3)^3 = a^9b^9 = ab^3$ and $(a^3b^3)^4 = a^{12}b^{12} = 1$, which shows that $M = \{a^3b^3, a^2, ab^3, 1\} \cong C_4$.

The powers of the generator element of N are a^2b^3 and $(a^2b^3)^2 = a^4b^6 = 1$, so $N = \{a^2b^3, 1\} \cong C_2$.

The order of $\bar{a}$ is the smallest k such that $a^k \in M$ or $a^k \in N$, respectively. So $o(\bar{a}) = 2$ in G/M and $o(\bar{a}) = 4$ in G/N .

On the other hand, $o(\bar{b}) = 6$ both in G/M and G/N because b, b^2, b^3, b^4, b^5 are not in M or N but $b^6 = 1$ is.

b) $G \cong C_4 \times C_6 \cong C_4 \times C_2 \times C_3$.

We have seen in part a) that $M \cong C_4$ and $M \cong C_2$.

$|G/M| = |G|/|M| = 4 \cdot 6/4 = 6$, and it is abelian, so $G/M \cong C_2 \times C_3$ (and it is also isomorphic to C_6 , but that is not its canonical form).

G/N is abelian with order $24/2 = 12$, so it can only be isomorphic to $C_4 \times C_3$ or $C_2 \times C_2 \times C_3$. But we have seen in part a) that G/N has an element of order 4 ($\bar{a}$ is such an element), so the second case is impossible. Thus $G/N \cong C_4 \times C_3$ (and it is also cyclic: $\cong C_{12}$).

HW1. Let $G = GL_2(\mathbb{Z}_3)$ and $g = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \in G$.

a) Determine the centralizer $C_G(g)$, that is, find all invertible matrices $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over $\mathbb{Z}_3$ such that $xg = gx$.

b) What is the size of the conjugacy class of g in G ?

HW2. What is the number of Abelian groups of order 600? What is the canonical form of the one among them which has the most elements of order 10?