LINEAR ALGEBRA AND ITS APPLICATIONS

Lukács Erzsébet, 2017

Assumed to be known (no review):

Gaussian elimination for solving linear systems of equations
matrix operations (including inversion)
determinant

Reviewed shortly:

vector spaces, linear maps and their matrices,
eigenvalues, eigenvectors, diagonalization

Introductory problem: There are n people sitting around a round table, everyone has a coin in front of them. They play the following game. Everyone checks the coin in front of their right side neighbour (at the same time) and if they see the head then they flip their own coins, if they see the tail then they don't do anything. They repeat this until all the coins show their tail sides. Which are those numbers n for which the game will end after a while, starting with any position of the coins?

Vector spaces, linear maps and matrices

Pl. $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathbb{R}[x]$, $\mathbb{C}[x]$, $C[0,1]$, etc.

V is a **vector space** over the field K

vectors: $\mathbf{u}, \mathbf{v}, \ldots \in V$,
scalars: $x, y, \alpha, \beta, \ldots, \lambda, \ldots \in K$,
operations: $\mathbf{u}+\mathbf{v} \in V$, $\lambda\mathbf{v} \in V$,
axioms: $\exists \mathbf{0}$, $\exists -\mathbf{v}$, identities.

K may be $\mathbb{R}$, $\mathbb{C}$, or other subfields of $\mathbb{C}$, or finite fields, e.g. for a prime p

$\mathbb{F}_p = \{\,0, 1, \ldots, p-1\,\}$, $+, \cdot$ modulo p.
Important: here $\alpha + \ldots + \alpha = n\alpha = 0$, if $p \mid n$,
$(\alpha+\beta)^p = \alpha^p + \beta^p$ (from the binomial theorem)

subspace: nonempty subset of V which is closed under the operations,
notation: $W \le V$ means that W is a subspace of V

e.g. the subspaces of $\mathbb{R}^3$ are: the origin, lines and planes containing the origin, and the whole $\mathbb{R}^3$
$\mathbb{R}[x] \ge \mathbb{R}[x]_{\le n}$: real polynomials of degree $\le n$

spanned subspace: the smallest subspace containing a given subset S

$=$ the intersection of all the subspaces containing S
$=$ the set of linear combinations of the elements of S, i.e.
$\{\,\sum \lambda_i \mathbf{v}_i \mid \mathbf{v}_i \in S,\ \lambda_i \in K\,\} =: \operatorname{span} S$

spanning set $\mathcal{S}$: spans the whole vector space, i.e. $\forall$ vector can be expressed as a linear combination of some elements of $\mathcal{S}$

linearly independent set $\mathcal{U} = \{\, \mathbf{u}_i \mid i \in I \,\}$: $\sum \lambda_i \mathbf{u}_i = \mathbf{0} \Rightarrow \lambda_i = 0\ \forall i$, i.e.
any vector in the spanned subspace can be written uniquely as a linear combination of elements from $\mathcal{U}$

basis: independent spanning set
= maximal independent set (no new elements can be added)
= minimal spanning set (no elements can be dropped)

$\forall$ independent set can be completed to a basis,
$\forall$ spanning set can be reduced to a basis

dimension the number of elements in a basis (well defined!)
The vector spaces in this course will be finite dimensional.

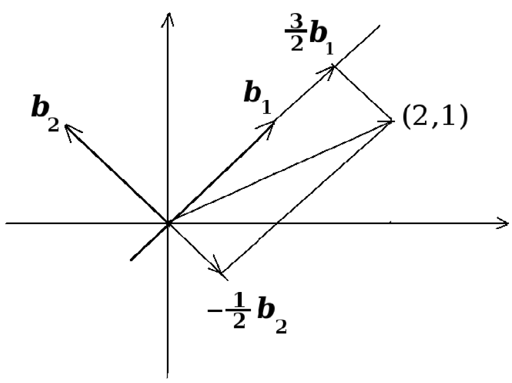
The following are equivalent for a set of vectors $\mathcal{B}$ in an n-dimensional space:
(*i*) $\mathcal{B}$ is a basis
(*ii*) $|\mathcal{B}| = n$, and $\mathcal{B}$ independent
(*iii*) $|\mathcal{B}| = n$, and $\mathcal{B}$ is a spanning set.

Example A basis (the standard basis) of $\mathbb{R}^{2\times 2}$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, the standard basis of $\mathbb{C}_{\mathbb{R}}$ is $\{\, 1, i \,\}$.

In an n-dimensional space with a basis $\mathcal{B} = \{\, \mathbf{b}_1, \ldots, \mathbf{b}_n \,\}$ (here the order of the elements is also important!), every vector can be uniquely written in the form $\sum_{i=1}^{n} x_i \mathbf{b}_i$. This defines the coordinatization with respect to $\mathcal{B}$: the **coordinate vector** of $\mathbf{v} = \sum x_i \mathbf{b}_i$ is

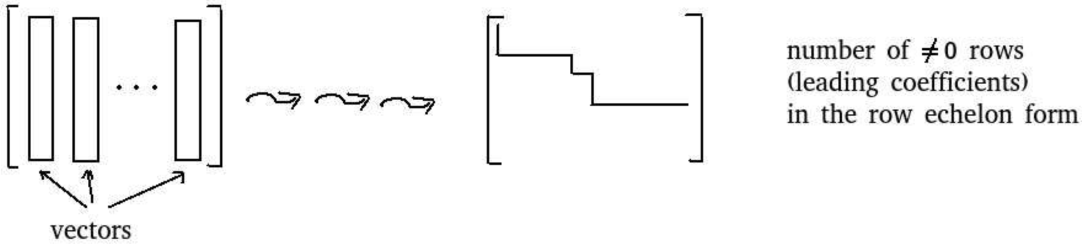
$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (x_1, \ldots, x_n)^T$$

Example In $\mathbb{R}^2$, what is $[(2,1)]_{\mathcal{B}}$ with respect to the basis $\mathcal{B} = \{(1,1), (-1,1)\,\}$?



$$[(2,1)]_{\mathcal{B}} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}.$$

rank (of a set of vectors): the dimension of the generated subspace.
calculation using Gauss elimination:



rank of a matrix: the dimension of the column space
= the dimension of the row space

linear map: $f : V \to W$ (V and W are vector spaces over K), which satisfies

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$
$$f(\lambda \mathbf{v}) = \lambda f(\mathbf{v})$$

Example: congruences of $\mathbb{R}^3$ fixing $\mathbf{0}$, differentiation in $\mathbb{R}[x]$.

linear transformation: linear map with $V = W$

matrix of a linear map:

$$f : V \to W$$

bases: $\mathcal{B}$ $\qquad$ $\mathcal{C}$

We need a matrix A such that $f : \mathbf{v} \mapsto \mathbf{w}$ if and only if $A \cdot [\mathbf{v}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{C}}$.
$\exists!$ such a matrix for $\mathcal{B}$ and $\mathcal{C}$:

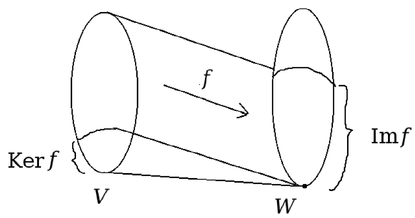
$$A = [f]_{\mathcal{B},\mathcal{C}} = \left[[f(\mathbf{b}_1)]_{\mathcal{C}} \;\middle|\; \ldots \;\middle|\; [f(\mathbf{b}_n)]_{\mathcal{C}} \right]$$

matrix of a linear transformation: usually $\mathcal{C} = \mathcal{B}$, and
$[f]_{\mathcal{B}} := [f]_{\mathcal{B},\mathcal{B}}$

Exercise: Determine the matrix of $z \to \overline{z}$ in $\mathbb{C}_{\mathbb{R}}$ in the basis $\{1, i\}$, or $\{i, 1+i\}$!

Sol.: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, or $\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$, respectively

image: $\operatorname{Im} f = \{ f(\mathbf{v}) \mid \mathbf{v} \in V \} \leq W$
kernel: $\operatorname{Ker} f = \{ \mathbf{v} \in V \mid f(\mathbf{v}) = \mathbf{0} \} \leq V$



Change of basis

Let $\mathcal{B} = \{\, \mathbf{b}_1, \ldots, \mathbf{b}_n \,\}$ and $\mathcal{B}' = \{\, \mathbf{b}'_1, \ldots, \mathbf{b}'_n \,\}$ be two bases in V. $P := \big[[\mathbf{b}'_1]_{\mathcal{B}} \big| \ldots \big| [\mathbf{b}'_n]_{\mathcal{B}}\big]$ is the transition matrix. Then

$[\mathbf{v}]_{\mathcal{B}} = P[\mathbf{v}]_{\mathcal{B}'}$, i.e. $P = [id]_{\mathcal{B}',\mathcal{B}}$, and
$P^{-1}[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}'}$.

Exercise: (a new method for an earlier problem) Determine the coordinate vector of $(2, 1)$ with respect to the basis $\{(1, 1), (-1, 1)\,\}$. This means that we change the standard basis $\mathcal{B} = \{(1, 0), (0, 1)\,\}$ to the new basis $\mathcal{B}' = \{(1, 1), (-1, 1)\,\}$.
The transition matrix is $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

$$[P|I] = \left[\begin{array}{rr|rr} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{rr|rr} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \mapsto \left[\begin{array}{rr|rr} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] = [I|P^{-1}].$$

$$[(2, 1)]_{\mathcal{B}'} = P^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

The matrix of a linear map with respect to a new pair of bases

Let the transition matrices from $\mathcal{B}$ to $\mathcal{B}'$ and from $\mathcal{C}$ to $\mathcal{C}'$ be P and Q, respectively, $[f]_{\mathcal{B},\mathcal{C}} = A$ and $[f]_{\mathcal{B}',\mathcal{C}'} = A'$.
Then $A' = Q^{-1}AP$:

$$[f(\mathbf{v})]_{\mathcal{C}'} \xleftarrow{Q^{-1}} [f(\mathbf{v})]_{\mathcal{C}} \xleftarrow{A} [\mathbf{v}]_{\mathcal{B}} \xleftarrow{P} [\mathbf{v}]_{\mathcal{B}'}$$

The matrix of a linear transformation with respect to a new basis

$\mathcal{B}$, $\mathcal{B}'$ are two bases of V, $f : V \to V$ a linear transformation, $[f]_{\mathcal{B}} = A$, $[f]_{\mathcal{B}'} = A'$, and P the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.
Then $A' = P^{-1}AP$.

Exercise: The matrix of the linear transformation $z \mapsto \overline{z}$ of $\mathbb{C}_{\mathbb{R}}$ with respect to the standard basis $\mathcal{B} = \{\, 1, i \,\}$ is $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. What is the matrix of the transformation with respect to the basis $\mathcal{B}' = \{\, i, 1 + i \,\}$?
The transition matrix is $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$, and the matrix of the transformation with respect to the new basis is $A' = P^{-1}AP = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$.

Definition. $A, B \in K^{n \times n}$ are **similar** (notation: $A \sim B$), if there is an invertible matrix P such that $B = P^{-1}AP$. In other words: A and B are the matrices of the same linear transformations in two bases (the columns of P give the new basis coordinatized in the old basis).

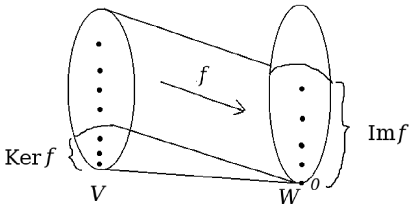
f **injective** if $\operatorname{Ker} f = \{\mathbf{0}\} =: 0$

f **surjective** if $\operatorname{Im} f = W$

f **isomorphism** if f injective és surjective.

Dimension theorem. Let $\dim V = n$ and $f : V \to W$ be linear. Then

$$\dim \operatorname{Ker} f + \dim \operatorname{Im} f = n$$



Cor.: If $f : V \to V$ and $\dim V = n$ then
f iso. $\Leftrightarrow$ f inj. $\Leftrightarrow$ f surj.

Example: the coordinatization is an isomorphism: for $|\mathcal{B}| = n$
$V \to K^n$
$\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$

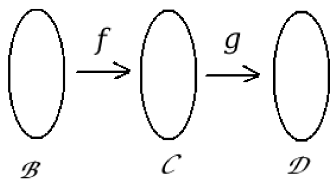
Theorem: Any map from the basis of a vector space to another vector space can be extended uniquely to a linear map.

rank of a linear map: $\operatorname{rank} f = \dim \operatorname{Im} f = \operatorname{rank}[f]_{\mathcal{B},\mathcal{C}}$ for any pair of bases $\mathcal{B}, \mathcal{C}$

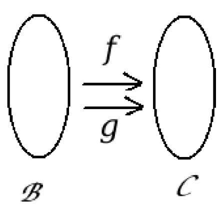
For $\dim V = n$, a linear map $f : V \to V$ is an isomorphism $\Leftrightarrow$ $\operatorname{rank} f = n$.

Matrix operations and linear maps:

$$[g]_{\mathcal{C},\mathcal{D}} \cdot [f]_{\mathcal{B},\mathcal{C}} = [g \circ f]_{\mathcal{B},\mathcal{D}}, \text{ where } (g \circ f)\mathbf{v} := g(f(\mathbf{v}))$$



$$[f]_{\mathcal{B},\mathcal{C}} + [g]_{\mathcal{B},\mathcal{C}} = [f + g]_{\mathcal{B},\mathcal{C}}, \text{ where } (f + g)(\mathbf{v}) := f(\mathbf{v}) + g(\mathbf{v})$$



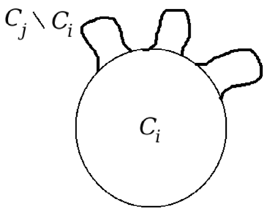
Proposition. *For the matrices A, B*
1) $\operatorname{rank}(AB) \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$
2) $|\operatorname{rank} A - \operatorname{rank} B| \leq \operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B$

Proof. Use the linear maps defined by the matrices. See Probelm Set 1.

An application: Fisher's inequality

Theorem. Ⓟ *Let $C_1, \ldots, C_k \subseteq \{1, \ldots, n\}$ be distinct sets. Suppose that there is a $\lambda > 0$ such that $|C_i \cap C_j| = \lambda$ $(\forall i \neq j)$. Then $k \leq n$.*

Proof. Case 1: $\exists i$: $|C_i| = \lambda$. Then:



$\Rightarrow n \geq |C_i| + (k-1) \geq k$.

Case 2: $\forall i$ $|C_i| = \lambda + a_i$, $a_i > 0$. The characteristic vector of $X \subseteq \{1, \ldots, n\}$ is the n dimensional 0-1-vector, $(x_1, \ldots, x_n)$, where $x_i = 1 \Leftrightarrow i \in X$. Let $M \in \mathbb{R}^{k \times n}$ the matrix whose ith row is the characteristic vector of the set C_i. Then

$$A = MM^T = \begin{bmatrix} \lambda + a_1 & \lambda & \lambda & \ldots & \lambda \\ \lambda & \lambda + a_2 & \lambda & \ldots & \lambda \\ \vdots & & \ddots & & \\ & & & & \lambda + a_n \end{bmatrix}_{k \times k}, \text{ since } \mathbf{x} \cdot \mathbf{y} = |X \cap Y|$$

We know: $\operatorname{rank} A \leq \operatorname{rank} M \leq n$.
We will show: $|A| \neq 0$, so $\operatorname{rank} A = k$.

$$|A| = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & \lambda + a_1 & \lambda & \ldots & \lambda \\ 0 & \lambda & \lambda + a_2 & \ldots & \lambda \\ \vdots & \vdots & & \ddots & \\ 0 & \lambda & \ldots & & \lambda + a_n \end{vmatrix}_{(k+1)\times(k+1)} = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ -\lambda & a_1 & 0 & \ldots & 0 \\ -\lambda & 0 & a_2 & \ldots & 0 \\ \vdots & & & \ddots & \\ -\lambda & 0 & 0 & \ldots & a_n \end{vmatrix} =$$

$$\begin{vmatrix} 1 + \frac{\lambda}{a_1} + \ldots + \frac{\lambda}{a_n} & 1 & 1 & \ldots & 1 \\ 0 & a_1 & 0 & \ldots & 0 \\ 0 & 0 & a_2 & \ldots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \ldots & a_n \end{vmatrix} = (1 + \frac{\lambda}{a_1} + \ldots + \frac{\lambda}{a_n}) \cdot a_1 \cdots a_n > 0,$$

since $\lambda, a_1, \ldots, a_n > 0$. $\square$

The rank of a matrix

For $A \in K^{m\times n}$ the following are equivalent:

(i) $\operatorname{rank} A = r$;
(ii) the rank of $\mathbf{x} \mapsto A\mathbf{x}$ is r;
(iii) the column space of A is r-dimensional;
(iv) the row space of A is r-dimensional;
(v) in the row echelon form of A there are exactly r nonzero rows (i.e. there are r leading coefficients);
(vi) A contains an $r\times r$ submatrix with nonzero determinant but all its $(r+1)\times(r+1)$ submatrices have zero determinant.

Invertible matrices

For $A \in K^{n\times n}$ the following are equivalent:

(i) A is invertible;
(ii) $f : K^n \to K^n$, $f : \mathbf{x} \mapsto A\mathbf{x}$ is an isomorphism;
(iii) $|A| \neq 0$;
(iv) the reduced row echelon form of A is I;
(v) $\operatorname{rank} A = n$;
(vi) the system of equations $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b} \in K^n$;
(vii) the system of equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

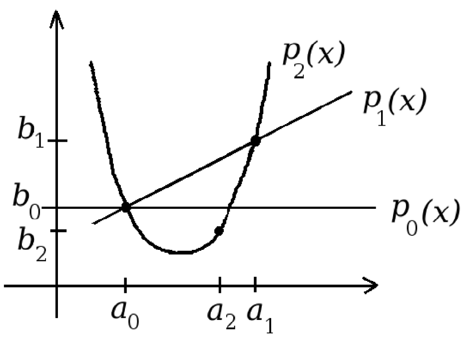
Calculating the inverse by Gaussian elimination:

$$[A|I] \mapsto\mapsto\mapsto [I|A^{-1}].$$

Polynomial interpolation

Ⓟ K is a field, $a_0, \ldots, a_n, b_0, \ldots, b_n \in K$, $a_0, \ldots, a_n$ are pairwise different $\Rightarrow$

$$\exists! p(x) \in K[x]_{\leq n} : \ p(a_i) = b_i \ \forall i.$$



Proof. $f : K[x]_{\leq n} \to K^{n+1}$, $f : p(x) \mapsto \begin{bmatrix} p(a_0) \\ \vdots \\ p(a_n) \end{bmatrix}$ is a linear map. $\operatorname{Ker} f = 0$, since if $p(x) \in \operatorname{Ker} f \Rightarrow p(a_0) = \cdots = p(a_n) = 0 \Rightarrow p(x) = (x-a_0)\cdots(x-a_n)q(x)$, but $\deg p \leq n$, so $p(x) = 0$. $\dim \operatorname{Ker} f + \dim \operatorname{Im} f = \dim K[x]_{\leq n} = n+1$ implies $\dim \operatorname{Im} f = n+1$, that is, f is surjective, and by $\operatorname{Ker} f = 0$ it is also injective, consequently, f is an isomorphism. This means that for any $\mathbf{b} = \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix}$ there is exactly one $p(x) \in K[x]_{\leq n}$ such that $f(p(x)) = \mathbf{b}$.

□

Newton's method of interpolation (see also the Lagrange polynomials)

For the given $a_0, \dots, a_n, b_0, \dots, b_n$ let $p_i(x) \in K[x]_{\leq i}$ be an interpolating polynomial on $a_0, \dots, a_i$. Clearly, $p_0(x) \equiv b_0$. If p_i is given, then

$$p_{i+1}(x) = p_i(x) + A \cdot (x - a_0) \cdots (x - a_i)$$

has the same values up to a_i for any $A \in K$, and $\deg p_{i+1}(x) \leq i+1$. Furthermore, A can be chosen so that $p_{i+1}(a_{i+1}) = b_{i+1}$ (if we substitute a_{i+1}, the coefficient of A is not 0, since all the a_j's are different). So in the end we find a suitable $p_n(x)$.

Remark: Using Newton's method, it is easy to improve an interpolation by adding new points, i.e. measuring the value of the function which we wish to approximate by a polynomial at a few more places.

Shamir's secret sharing

We want to share a secret between n people (let the secret be coded by a natural number c) so that any k of the n people together can find out the secret information, but no $k-1$ of them could get closer to the secret if they share their bit of information among them.

Solution: Let $p > c$ be a prime, $q(x) \in \mathbb{F}_p[x]_{<k}$, such that $q(0) = c$ (that is, c is the constant term). The i.'th person is given the value $q(i) \in \mathbb{F}_p$ $(i = 1, \dots, n)$. Then k people together know k values of the polynomial, so by the interpolation theorem they can determine the polynomial and then also its constant term. But if someone knows only $k-1$ values of the polynomial, then $q(0)$ can still be anything: we can still find such an interpolating polynomial of degree less than k.

Question: Why do we need a polynomial over a finite field $\mathbb{F}_p$? Why do not we choose an integral polynomial? Because in that case it is not true that with given $k-1$ values, $q(0)$ can be anything. It is possible that, though we find an interpolating polynomial over $\mathbb{Q}$, the coefficients of that polynomial are not integers.