1. Which of the following sets form a vector space over $\mathbb{R}$? Give a basis of the vector spaces.
a) 3×3 real upper triangular matrices with the usual operations;
b) invertible 2×2 real matrices;
c) polynomials of degree at most 4 which have -1 as one of their roots;
d) real pairs with addition $(a, b) \oplus (c, d) = (a + d, b + c)$ and multiplication by scalars $\lambda \cdot (a, b) = (\lambda a, \lambda b)$.

2. Determine the matrices of the following linear maps with respect to the given basis or pair of bases:
a) rotation of the 3 dimensional space about the z axis by $90°$, in the standard basis;
b) $p(x) \mapsto (xp(x))'$ in the space of real polynomials of degree at most 2, in the standard basis $\{1, x, x^2\}$;
c) $\mathbf{x} \mapsto A\mathbf{x}$, where $A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$, $\mathcal{B} = \{(1, 2), (1, 1)\}$;
d) $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, where $\varphi(1, 2, 1) = (0, 2, 1)$, $\varphi(1, 1, 1) = (1, 0, 0)$, $\varphi(1, 0, 0) = (-1, 0, 0)$, in the standard basis;
e) $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(x, y) = (x + y, y, x)$, in the pair of bases $\mathcal{B}_1 = \{(1, 1), (2, 0)\}$, $\mathcal{B}_2 = \{(1, 2, 1), (-1, 1, 0), (0, 1, 1)\}$;
f) orthogonal projection onto the plane $x - 2y + z = 0$, in the standard basis;
g) transposition of 2×2 real matrices, in the standard basis.

3. Find a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that
a) $0 \neq \operatorname{Ker} f \subseteq \operatorname{Im} f$;
b) $\operatorname{Ker} f$ is 1 dimensional, and $\operatorname{Ker} f \cap \operatorname{Im} f = \{\mathbf{0}\}$;
c) $\operatorname{Im} f$ is 2 dimesional, and f maps each vector of $\operatorname{Im} f$ into itself;
d) $f^3 = 0$ but $f^2 \neq 0$ (where the product means composition).

4. Let A be the standard matrix of $f : (x, y, z) \longmapsto (x + y - 2z, x + z, 2x + y - z, -x - z)$. Give bases for the null space of A (i.e. the kernel of f) and for the columns space of A (i.e. the image of f).

5. Prove that
a) $\operatorname{rank}(AB) \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$, where $A \in K^{k \times m}$ és $B \in K^{m \times n}$;
b) $|\operatorname{rank} A - \operatorname{rank} B| \leq \operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B$, where $A, B \in K^{m \times n}$.
(*Hint: Prove that, considering the matrices as linear maps in the natural way,* $\operatorname{Im} AB \leq \operatorname{Im} A$, $\operatorname{Ker} AB \geq \operatorname{Ker} B$ *and* $\operatorname{Im}(A + B) \leq \operatorname{span}(\operatorname{Im} A,\ \operatorname{Im} B)$.)

6. Show that for any matrix $A \in K^{m \times n}$ and any invertible matrices $B \in K^{m \times m}$ and $C \in K^{n \times n}$, we have $\operatorname{rank} BA = \operatorname{rank} AC = \operatorname{rank} A$.

7. Show that for every matrix $A \in K^{m \times n}$ of rank r there exist invertible matrices $P \in K^{n \times n}$ and $Q \in K^{m \times m}$ such that in the matrix $B = Q^{-1}AP$ the elements $b_{11}, \ldots, b_{rr}$ are 1, and all the other elements are 0, i.e. as a block matrix $B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

8. Let f be a linear transformation of a 6 dimensional vector space. Which of the following sequences may give the ranks of f, f^2, f^3, f^4?
a) $3, 4, 2, 2$ b) $6, 5, 4, 3$ c) $5, 4, 4, 4$ d) $5, 3, 2, 1$ e) $3, 2, 1, 0$

9. Show that every 3×3 real matrix has an eigenvector.

10. Prove that every eigenvector of A is an eigenvector of A^2. Is the reverse statement true?

11. Which are those $n \times n$ real matrices for which every nonzero element of $\mathbb{R}^n$ is an eigenvector?

12. Determine the eigenvalues and eigenvectors of the linear transformations in problem 2.

13. For which intergers c is there an integral polynomial $f(x)$ with $f(1) = 0$, $f(2) = 2$ és $f(0) = c$?