**1.** Write the vector $\mathbf{b}$ as the sum of a vector which is orthogonal to $\mathbf{a}$ and a vector which is parallel to $\mathbf{a}$ if

a) $\mathbf{a} = (1, -2, 0, 1)$, $\mathbf{b} = (3, 1, 1, 1)$;

b) $\mathbf{a} = (1 + i, 1 - i)$, $\mathbf{b} = (i, 3 - i)$.

**2.** Suppose $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathbb{R}^n$ are orthogonal vectors and neither of them is $\mathbf{0}$. Let $W = \operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$, and for a vector $\mathbf{v} \in \mathbb{R}^n$ define $\mathbf{v}' = \sum_{i=1}^{k} \frac{\mathbf{b}_i^T \mathbf{v}}{|\mathbf{b}_i|^2} \mathbf{b}_i$. Prove that

a) $\mathbf{v} - \mathbf{v}' \perp \mathbf{w}$ for every $\mathbf{w} \in W$;

b) if $\mathbf{v} \notin W$, then $\{\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{v} - \mathbf{v}'\}$ is an orthogonal basis of $\operatorname{span}(W \cup \{\mathbf{v}\})$;

c) $\mathbf{v}'$ is the element of $W$ closest to $\mathbf{v}$ (that is, $|\mathbf{v} - \mathbf{v}'| = \min\{|\mathbf{v} - \mathbf{w}| \mid \mathbf{w} \in W\}$).

**3.** Use problem 2.b) to find and orthogonal and then an orthonormal basis in the subspace of $\mathbb{R}^4$ spanned by $(1, 2, -1, 0)$, $(2, 1, 0, 1)$ and $(1, -1, 1, -1)$.

**4.** Prove that the subset $\{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 = x_4 + x_5\}$ is a hyperplane in $\mathbb{R}^5$, and determine its normal vector. Calculate the reflection of $(1, 0, 0, 0, 0)$ to this hyperplane.

**5.** Give the standard matrix of the orthogonal projection and of the reflection on the hyperplane $x + y - z = 0$ in $\mathbb{R}^3$.

**6.** Find the standard matrix of a reflection which maps the vector $(1, 2, -2)$ to $(3, 0, 0)$. (Hint: It is the reflection on the bisector plane of the line segment connecting the endpoints of the two vectors.)

**7.** Which of the following matrices are self-adjoint, unitary or normal? Which of the self-adjoint matrices are positive semidefinite or positive definite?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} \quad C = \begin{bmatrix} i & i \\ i & -i \end{bmatrix} \quad D = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \quad F = \begin{bmatrix} -1 & 2+i \\ 2-i & -5 \end{bmatrix} \quad G = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix} \quad H = \begin{bmatrix} 1 & i \\ 1+i & 0 \end{bmatrix}$$

**8.** Give an example of a transformation $\mathbb{C}^n \to \mathbb{C}^n$ such that the absolute value of every eigenvalue is 1 but the transformation is not unitary.

**9.** Give the reduced (and the full) singular value decomposition of the following matrices.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -3 \end{bmatrix} \quad C = \begin{bmatrix} i & 0 \\ 0 & 1 \\ 0 & i \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \quad E = \begin{bmatrix} 2 & -11 \\ 10 & -5 \end{bmatrix}$$

**10.** Use the reduced SVD form of the matrices of problem 9 to

a) find the pseudoinverse of $A$, and with that the best approximate solution of the inconsistent equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$;

b) find the matrix of rank 1 closest to the matrix $D$.