1. *Which of the following sets form a vector space over $\mathbb{R}$? Give a basis of the vector spaces.*

a) 3×3 *real upper triangular matrices with the usual operations;*
b) invertible 2×2 *real matrices;*
c) polynomials of degree at most 4 *which have* -1 *as one of their roots;*
d) real pairs with addition $(a,b)\oplus(c,d)=(a+d,b+c)$ *and multiplication by scalars* $\lambda\cdot(a,b)=(\lambda a,\lambda b)$.

Solution: a) It is a vector space because it is a subspace of $\mathbb{R}^{3\times 3}$: the sum or scalar multiple of upper triangular matrices is upper triangular. A basis is $\{\, E_{ij} \mid i\le j\,\}$, where E_{ij} is the matrix where the j'th element of the ith row is 1, and the others are 0.

b) Not a vector space. Though it is a subset of the vector space $\mathbb{R}^{2\times 2}$, it is not a subspace: 0 times an invertible matrix is the zero matrix, which is not invertible. But even if we add the 0 matrix to the subset we do not get a subspace: $\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ is not invertible.

c) It is a vector space because it is a subspace of $\mathbb{R}[x]$ (the sum and scalar multiples of polynomials with root -1 also have -1 as a root and the degree does not increase). A basis is: $\{\, x+1,\ (x+1)x,\ (x+1)x^2,\ (x+1)x^3\,\}$.

d) Not a vector space: $(a,b)\oplus(c,d)=(a+d,b+c)$, but $(c,d)\oplus(a,b)=(c+b,a+d)$, so the operation $\oplus$ is not commutative.

2. *Determine the matrices of the following linear maps with respect to the given basis or pair of bases:*

a) rotation of the 3 *dimensional space about the* z *axis by* $90°$*, in the standard basis;*
b) $p(x)\mapsto (xp(x))'$ *in the space of real polynomials of degree at most* 2*, in the standard basis* $\{\,1,x,x^2\,\}$;
c) $\mathbf{x}\mapsto A\mathbf{x}$*, where* $A=\begin{bmatrix}1&-1\\4&-3\end{bmatrix}$, $\mathcal{B}=\{(1,2),(1,1)\,\}$;
d) $\varphi:\mathbb{R}^3\to\mathbb{R}^3$*, where* $\varphi(1,2,1)=(0,2,1)$, $\varphi(1,1,1)=(1,0,0)$, $\varphi(1,0,0)=(-1,0,0)$*, in the standard basis;*
e) $\varphi:\mathbb{R}^2\to\mathbb{R}^3$, $\varphi(x,y)=(x+y,y,x)$*, in the pair of bases* $\mathcal{B}_1=\{(1,1),(2,0)\,\}$, $\mathcal{B}_2=\{(1,2,1),(-1,1,0),(0,1,1)\,\}$;
f) orthogonal projection onto the plane $x-2y+z=0$*, in the standard basis;*
g) transposition of 2×2 *real matrices, in the standard basis.*

Solution: a) $\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$ b) $\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$ c) $\begin{bmatrix}1&1\\2&1\end{bmatrix}^{-1}\begin{bmatrix}1&-1\\4&-3\end{bmatrix}\begin{bmatrix}1&1\\2&1\end{bmatrix}=\begin{bmatrix}-1&1\\0&-1\end{bmatrix}$

d) $A\cdot\begin{bmatrix}1&1&1\\2&1&0\\1&1&0\end{bmatrix}=\begin{bmatrix}0&1&-1\\2&0&0\\1&0&0\end{bmatrix}\ \Rightarrow\ A=\begin{bmatrix}0&1&-1\\2&0&0\\1&0&0\end{bmatrix}\cdot\begin{bmatrix}1&1&1\\2&1&0\\1&1&0\end{bmatrix}^{-1}=\begin{bmatrix}-1&-1&3\\0&2&-2\\0&1&-1\end{bmatrix}$

e) $\begin{bmatrix}1&-1&0\\2&1&1\\1&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&1\\0&1\\1&0\end{bmatrix}\begin{bmatrix}1&2\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\-1&-2\\0&2\end{bmatrix}$

f) The projection of the vector (x,y,z) onto the normal vector $(1,-2,1)$ of the plane is $\dfrac{(x,y,z)(1,-2,1)}{|(1,-2,1)|^2}(1,-2,1)=\dfrac{1}{6}\cdot(x-2y+z,\ -2x+4y-2z,\ x-2y+z)$, so its projection onto the plane is
$(x,y,z)-\frac{1}{6}(x-2y+z,\ -2x+4y-2z,\ x-2y+z)=(\frac{1}{6}(5x+2y-z,\ 2x+2y+2z,\ -x+2y+5z)$,
and the standard matrix of the projection is $\frac{1}{6}\begin{bmatrix}5&2&-1\\2&2&2\\-1&2&5\end{bmatrix}$.

g) The action on the elements of the standard basis $\{\,E_{11},E_{12},E_{21},E_{22}\,\}$ is $E_{11}\mapsto E_{11}$, $E_{12}\mapsto E_{21}$, $E_{21}\mapsto E_{12}$, $E_{22}\mapsto E_{22}$, so the standard matrix is $\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$.

3. *Find a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that*
a) $0 \neq \operatorname{Ker} f \subseteq \operatorname{Im} f$;
b) $\operatorname{Ker} f$ *is* 1 *dimensional, and* $\operatorname{Ker} f \cap \operatorname{Im} f = \{\mathbf{0}\}$;
c) $\operatorname{Im} f$ *is* 2 *dimesional, and* f *maps each vector of* $\operatorname{Im} f$ *into itself;*
d) $f^3 = 0$ *but* $f^2 \neq 0$ *(where the product means composition).*

Solution: It suffices the give the action of the transformations on the elements of a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathbb{R}^3$.

a) By the dimension theorem, $\dim \operatorname{Ker} f + \dim \operatorname{Im} f = 3$, and $1 \leq \dim \operatorname{Ker} f \leq \dim \operatorname{Im} f$, so $\dim \operatorname{Ker} f = 1$, and $\dim \operatorname{Im} f = 2$. This can be achieved by a linear map acting on the basis elements as follows: $\mathbf{b}_1 \mapsto \mathbf{0}$, $\mathbf{b}_2 \mapsto \mathbf{b}_1$, $\mathbf{b}_3 \mapsto \mathbf{b}_3$

b) Let $\mathbf{b}_1 \mapsto \mathbf{0}$, $\mathbf{b}_2 \mapsto \mathbf{b}_2$ and $\mathbf{b}_3 \mapsto \mathbf{b}_3$. Since the image is a 2-dimensional subspace $\operatorname{span}(\mathbf{b}_2, \mathbf{b}_3)$, the dimension of the kernel can only be 1, and the kernel contains $\mathbf{b}_1$, so it is the subspace $\operatorname{span}(\mathbf{b}_1)$.

c) The map given in the solution of b) satisfies this condition as well.

d) $\mathbf{b}_1 \mapsto \mathbf{b}_2 \mapsto \mathbf{b}_3 \mapsto \mathbf{0}$

4. *Let A be the standard matrix of $f : (x, y, z) \longmapsto (x + y - 2z, x + z, 2x + y - z, -x - z)$. Give bases for the null space of A (i.e. the kernel of f) and for the columns space of A (i.e. the image of f).*

Solution: The matrix and its reduced row echelon form are

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \mapsto \mapsto \mapsto \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the null space, i.e. the solution space of the homogeneous system of equations consists of the matrices $t \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ $(t \in \mathbb{R})$, so its basis is $\{ [-1 \;\; 3 \;\; 1]^T \}$. The basis of the column space consists of the columns of A which stand in the same positions as the columns of $\operatorname{rref}(A)$ containing a leading 1, i.e. $\{ [1 \;\; 1 \;\; 2 \;\; -1]^T, [1 \;\; 0 \;\; 1 \;\; 0]^T \}$. We could also find a basis of the row space of A easily: the nonzero rows of $\operatorname{rref}(A)$, i.e. $\{ [1 \;\; 0 \;\; 1], [0 \;\; 1 \;\; -3] \}$ give such a basis, since they are clearly linearly independent, and the elementary row operations do not change the row space.

5. *Prove that*
a) $\operatorname{rank}(AB) \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$, *where* $A \in K^{k \times m}$ *és* $B \in K^{m \times n}$;
b) $|\operatorname{rank} A - \operatorname{rank} B| \leq \operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B$, *where* $A, B \in K^{m \times n}$.
(Hint: Prove that, considering the matrices as linear maps in the natural way, $\operatorname{Im} AB \leq \operatorname{Im} A$, $\operatorname{Ker} AB \geq \operatorname{Ker} B$ *and* $\operatorname{Im}(A + B) \leq \operatorname{span}(\operatorname{Im} A, \operatorname{Im} B)$.*)*

Solution: For simplicity, we also denote by A and B the natural linear map corresponding to the matrices A and B, respectively: $A : \mathbf{x} \mapsto A\mathbf{x}$ and $B : \mathbf{x} \mapsto B\mathbf{x}$.

a) $(AB)\mathbf{x} = A(B\mathbf{x}) \in \operatorname{Im} A$, so $\operatorname{Im} AB \leq \operatorname{Im} A$, thus $\operatorname{rank} AB \leq \operatorname{rank} A$. On the other hand, if $B\mathbf{x} = \mathbf{0}$, then $(AB)\mathbf{x} = A\mathbf{0} = \mathbf{0}$, so $\operatorname{Ker} B \leq \operatorname{Ker} AB$, which implies by the dimension theorem that $\operatorname{rank} AB \leq \operatorname{rank} B$. The two inequalities together prove the statement.

b) $\{(A + B)\mathbf{x} \mid \mathbf{x} \in K^n\} = \{A\mathbf{x} + B\mathbf{x} \mid \mathbf{x} \in K^n\} \leq \operatorname{Im} A + \operatorname{Im} B$. In general, $\dim(U + W) \leq \dim U + \dim W$ for the subspaces $U, W \leq V$, since the union of the bases of U and W clearly spans $U + W := \operatorname{span}(U, W)$, and its maximal independent subset will be a basis of $U + W$. This gives in our case that $\operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B$. If we apply this for $A+B$ and $-B$, we get $\operatorname{rank} A = \operatorname{rank}((A + B) + (-B)) \leq \operatorname{rank}(A + B) + \operatorname{rank}(-B) = \operatorname{rank}(A + B) + \operatorname{rank} B$, implying that $\operatorname{rank}(A + B) \geq \operatorname{rank} A - \operatorname{rank} B$, and similarly, $\operatorname{rank}(A + B) \geq \operatorname{rank} B - \operatorname{rank} A$. The two together gives that $\operatorname{rank}(A + B) \geq |\operatorname{rank} A - \operatorname{rank} B|$.

6. *Show that for any matrix $A \in K^{m \times n}$ and any invertible matrices $B \in K^{m \times m}$ and $C \in K^{n \times n}$, we have* $\operatorname{rank} BA = \operatorname{rank} AC = \operatorname{rank} A$.

Solution: Problem 5.a) implies that $\operatorname{rank} BA \le \operatorname{rank} A$, and $\operatorname{rank} A = \operatorname{rank} B^{-1}(BA) \le \operatorname{rank} BA$, so $\operatorname{rank} BA = \operatorname{rank} A$, and similarly, $\operatorname{rank} AC = \operatorname{rank} A$.

7. *Show that for every matrix $A \in K^{m\times n}$ of rank r there exist invertible matrices $P \in K^{n\times n}$ and $Q \in K^{m\times m}$ such that in the matrix $B = Q^{-1}AP$ the elements $b_{11}, \dots, b_{rr}$ are 1, and all the other elements are 0, i.e. as a block matrix $B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.*

Solution: Version 1: We can transform the matrix A to its reduced row echelon form by elementary row operations (every elementary row operation is a left multiplication by an invertible matrix, so the sequence of row operations will be a left multiplication by the product of these matrices, which is also invertible. Q can be the inverse of this matrix). Then we do elementary column operations: we move the columns containing leading ones to the left, then use these to make all the other columns zero. Every elementary column operation is a right multiplication by an invertible matrix, so the sequence of these operations can be performed by the right multiplication by the product of these matrices, which is also invertible (let it be P). In the end we obtained $Q^{-1}AP$, which is of the requested form.

Version 2: The statement is equivalent to saying that we can find a suitable pair of bases such that the matrix with respect to that pair is in the requested form. Let $\mathcal{B}_2$ be a basis of $\operatorname{Ker} A$, and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ a basis of K^n. Furthermore, let $\mathcal{C}_1 = \{ A\mathbf{x} \mid \mathbf{x} \in \mathcal{B}_1 \}$. We show that $\mathcal{C}_1$ is linearly independent. Indeed, if $\sum_{i=1}^{r} \lambda_i A\mathbf{b}_i = \mathbf{0}$ (where $\mathcal{B}_1 = \{ \mathbf{b}_i \mid i = 1, \dots, r \}$), then $A(\sum_{i=1}^{r} \lambda_i \mathbf{b}_i) = \mathbf{0}$, i.e. $\sum_{i=1}^{r} \lambda_i \mathbf{b}_i \in \operatorname{Ker} A$, but the basis of $\operatorname{Ker} A$ (i.e. $\mathcal{B}_2$) is independent from $\mathcal{B}_1$, so this implies that $\lambda_i = 0$ for $i = 1, \dots, r$. If we now choose a basis $\mathcal{C}$ of K^m which extends the set C_1: $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, then the matrix of A in the pair of bases $\mathcal{B}, \mathcal{C}$ is the block matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Since the latter matrix clearly has rank r, and it can be written as $Q^{-1}AP$ with some invertible transition matrices P and Q, $\operatorname{rank} A = r$ by problem 6.

8. *Let f be a linear transformation of a 6 dimensional vector space. Which of the following sequences may give the ranks of f, f^2, f^3, f^4?*

a) $3,4,2,2$ *b)* $6,5,4,3$ *c)* $5,4,4,4$ *d)* $5,3,2,1$ *e)* $3,2,1,0$

Solution: Let us observe first that $\operatorname{Im} id \ge \operatorname{Im} f \ge \operatorname{Im} f^2 \ge \operatorname{Im} f^3 \ge \cdots$, so by using the notation $r_i = \operatorname{rank} f_i$ (with $r_0 := \operatorname{rank} id = \dim V$), we get

$$r_0 \ge r_1 \ge r_2 \ge \cdots \qquad (*)$$

for every linear transformation $f : V \to V$. Furthermore, $\operatorname{Im} f^n = \{ f^n(\mathbf{v}) \mid \mathbf{v} \in V \} = \{ f(f^{n-1}(\mathbf{v})) \mid \mathbf{v} \in V \} = \operatorname{Im} f|_{\operatorname{Im} f^{n-1}}$, where $f|_U$ is the restriction of f to the subspace U, i.e. $f_U : U \to V$ acts on the element of U the same way as f. It follows from the dimension theorem that $r_n = \dim \operatorname{Im} f|_{\operatorname{Im} f^{n-1}} = r_{n-1} - \dim \operatorname{Ker} f|_{\operatorname{Im} f^{n-1}}$, i.e. $r_{n-1} - r_n = \dim \operatorname{Ker} f|_{\operatorname{Im} f^{n-1}} = \dim(\operatorname{Ker} f \cap \operatorname{Im} f|_{\operatorname{Im} f^{n-1}})$. Since the latter form a decreasing sequence, we get that

$$r_0 - r_1 \ge r_1 - r_0 \ge r_2 - r_1 \ge \cdots \qquad (**).$$

By $(*)$, the sequence in a) cannot be a sequence of ranks, and by $(**)$ b) and d) are also impossible. For c) and e) we can give suitable transformations.

c) $\mathbf{b}_1 \mapsto \mathbf{b}_2 \mapsto \mathbf{0}$, $\mathbf{b}_i \mapsto \mathbf{b}_i$, if $i = 3, 4, 5, 6$.

e) $\mathbf{b}_1 \mapsto \mathbf{0}$, $\mathbf{b}_2 \mapsto \mathbf{0}$, $\mathbf{b}_3 \mapsto \mathbf{b}_4 \mapsto \mathbf{b}_5 \mapsto \mathbf{b}_6 \mapsto \mathbf{0}$.

9. *Show that every 3×3 real matrix has an eigenvector.*

Solution: The characteristic polynomial $-x^3 + a_2x^2 + a_1x + a_0$ must have at least one real root (since its limit in $-\infty$ is $+\infty$, and in $+\infty$ is $-\infty$). So the matrix has a real eigenvalue λ, and for the real eigenvalue we can find real eigenvectors by solving the system of equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$ over $\mathbb{R}$.

10. *Prove that every eigenvector of A is an eigenvector of A^2. Is the reverse statement true?*

Solution: If $\mathbf{v}$ is an eigenvector for the eigenvalue λ, then $A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$. The reverse statement is usually not true, for example the rotation of the plane about the origin by 90° has no real eigenvector, while for its square every nonzero vector is an eigenvector.

11. *Which are those $n \times n$ real matrices for which every nonzero element of $\mathbb{R}^n$ is an eigenvector?*

Solution: We show that only the scalar matrices λI have this property. If A had eigenvectors $\mathbf{u}$ and $\mathbf{v}$ corresponding to two different eigenvalues λ and μ, respectively, then these were independent (if one were a scalar multiple of the other then the corresponding eigenvalues would be the same). But by the condition, $\mathbf{u}+\mathbf{v}$ would also be an eigenvector: $A(\mathbf{u}+\mathbf{v}) = \nu\mathbf{u}+\nu\mathbf{v}$. On the other hand, $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = \lambda\mathbf{u}+\mu\mathbf{v}$, so $\nu\mathbf{u}+\nu\mathbf{v} = \lambda\mathbf{u}+\mu\mathbf{v}$, that is, $(\nu-\lambda)\mathbf{u}+(\nu-\mu)\mathbf{v} = \mathbf{0}$, and by the linear independence, $\nu-\lambda = \nu-\mu = 0$, i.e. $\lambda = \nu = \mu$, contradicting the assumption.

12. *Determine the eigenvalues and eigenvectors of the linear transformations in problem 2.*

Solution: a) The real eigenvectors are only those nonzero vectors that are parallel to the z axis, the eigenvalue corresponding to them is 1.

b) It can be seen immediately from the matrix that its eigenvalues are $1, 2, 3$, and the corresponding eigenvectors are the nonzero scalar multiples of the basis elements, i.e. the polynomials a, bx, and cx^2 $(a, b, c \neq 0)$.

c) The characteristic polynomial is $|A - xI| = x^2 + 2x + 1$, which has only one root, -1 (with multiplicity 2). The eigenvectors corresponding to the eigenvalue -1 are the nonzero solutions of the equation $(A + I)\mathbf{v} = \mathbf{0}$, i.e. $t \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, where $t \neq 0$.

d) For the matrix A in the solution of 2.d), the characteristic polynomial is $|A - xI| = -x(x-1)(x+1)$, so the eigenvalues are $\lambda = 0, -1, 1$, and the corresponding eigenvectors can be determined by solving the systems of equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
The results are: $t \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $t \cdot \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$, and $t \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, respectively.

e) This is not a linear transformation, because it maps a vector space to a different space (actually, to a space whose dimension is also different from that of the first space).

f) We can deduce from the geometric meaning that the eigenvectors are nonzero vectors of the plane with eigenvalue 1 (so the plane is a 2-dimensional eigenspace), and the normal vectors of the plane (the nonzero scalar multiples of $[1 \ \ -2 \ \ 1]^T$) with eigenvalue 0.

g) The eigenvectors are the nonzero symmetric matrices (of the form $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$) with eigenvalue 1, and the nonzero skew symmetric matrices (of the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$) with eigenvalue -1. There cannot be any other real eigenvalues, because the eigenspaces are independent and the sum of their dimensions is 4, the dimension of the whole space $\mathbb{R}^{2\times 2}$.

13. *For which intergers c is there an integral polynomial $f(x)$ with $f(1) = 0$, $f(2) = 2$ és $f(0) = c$?*

Solution: Let us calculate the interpolating polynomial over $\mathbb{Q}$. For the condition $f(1) = 0$ we get $p_1(x) \equiv 0$. If we add the new condition $f(2) = 2$, we see that $p_2(x) = 0 + (x-1)a$, where $2 = p_2(2) = a$, so $a = 2$ and $p_2(x) = 2x - 2$. Finally, adding the third condition, we get that $p_3(x) = 2x - 2 + b(x-1)(x-2)$, and $c = p_3(0) = -2 + 2b$, so $b = \frac{c}{2} + 1$, and $p_3(x) = 2x - 2 + (\frac{c}{2} + 1)(x-1)(x-2) = (\frac{c}{2} + 1)x^2 + (-1 - \frac{3c}{2})x + c$, which has integer coefficients if c is an even integer.
If c is odd, then the second conditions contradicts the third when the polynomial has integer coefficients: if we substitute 2 and 0, the value of the polynomial should be the same modulo 2. (or we can simply consider the integral polynomial as a polynomial over $\mathbb{F}_2$). So there is an integral polynomial with the given values if and only if c is even.