1. *Is there a 3×3 matrix over $\mathbb{Q}$ with minimal polynomial*

a) $x^2 - 2$;

b) $x^2 + x$?

Solution: a) If $m_A(x) = x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$, then the only eigenvalues of A are $\sqrt{2}$ and $-\sqrt{2}$. But then $k_A(x) = -(x - \sqrt{2})^{a_1}(x + \sqrt{2})^{a_2}$, where $a_1, a_2 \geq 1$, and $a_1 + a_2 = 3$. So $k_A(x)$ is either $-m(x)(x - \sqrt{2}) = -(x^2 - 2)(x - \sqrt{2}) = -x^3 + \sqrt{2}x^2 + 2x - 2\sqrt{2}$ or $k_A(x) = -m(x)(x + \sqrt{2}) = -(x^2 - 2)(x + \sqrt{2}) = -x^3 - \sqrt{2}x^2 - 2x + 2\sqrt{2}$, but neither of them is in $\mathbb{Q}[x]$. So there is no such matrix.

b) Since $m_A(x) = x^2 + x = x(x + 1)$, the eigenvalues are 0 and -1. Furthermore, both of them have multiplicity 1 in the minimal polinomial, so the matrix A is diagonalizable. This condition is satisfied by an arbitrary diagonal matrix which has only 0s and -1s in its diagonal. Actually every 2×2 matrix with minimal polynomial $x^2 + x$ is similar to one of the diagonal matrices with diagonal elements $0, 0, -1$ or $0, -1, -1$.

2. *Suppose that A is a matrix over $\mathbb{C}$ such that $A^m = I$ for some $m \geq 1$. Prove that A is diagonalizable.*

Solution: Since A is a 'root' of the polynomial $x^m - 1$, the minimal polynomial is a divisor of $x^m - 1$. But $x^m - 1$ has m different roots in $\mathbb{C}$ (the m complex mth roots of unity), so the minimal polynomial has no multiple roots. Thus the matrix A is diagonalizable.

3. *Which of the following matrices are diagonalizable over $\mathbb{C}$? Determine the Jordan normal form of the matrices.*

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: The characteristic polynomial of the matrix A is $-x^3 - 3x^2 + 6x = -x(x + \frac{3}{2} - \frac{\sqrt{33}}{2})(x + \frac{3}{2} + \frac{\sqrt{33}}{2})$. A is diagonalizable because it has 3 different eigenvalues. The Jordan normal form in this case is any of the diagonal matrices with the eigenvalues 0, $-\frac{3}{2} + \frac{\sqrt{33}}{2}$ and $-\frac{3}{2} - \frac{\sqrt{33}}{2}$ in its diagonal, in some order.

$k_B(x) = -x^3 + 3x - 2 = -(x - 1)^2(x + 2)$, so the eigenvalues are 1 and -2. The eigenspace corresponding to the eigenvalue 1 is the solution space of the equation $(B - I)\mathbf{x} = \mathbf{0}$, but it is 1-dimensional, so the Jordan normal form has only one 1-block, whose size is 2, and one -2-block of size 1. Since the Jordan normal form is not diagonal, the matrix is not diagonalizable.

C is a triangular matrix, so without any further calculation we can see that its eigenvalues are the diagonal elements $1, 2, 3$ and 4. Since C is a 4×4 matrix with four different eigenvalues, C must be diagonalizable, and its Jordan normal form is the diagonal matrix with $1, 2, 3, 4$ in the diagonal.

$k_D(x) = x^4$, so 0 is the only eigenvalue. The Jordan normal form of D contains only Jordan blocks corresponding to 0. The rank of D is 2, thus the eigenspace corresponding to 0 is $4 - 2 = 2$-dimensional, consequently the Jordan normal form has only two Jordan blocks. $D^2 = 0$, so the largest Jordan block is of size 2. Hence the Jordan normal form is the block diagonal matrix with two diagonal blocks, each equal to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

4. *What is the maximal number of non-similar complex matrices satisfying the following conditions? Give the Jordan normal form in each possible case.*

a) $k(x) = -x^5(x + 1)^2$, $m(x) = x^3(x + 1)$;

b) $k(x) = (x - 1)^4 x$, *and the eigenspace for the eigenvalue* 1 *is 2-dimensional.*

Solution: a) We know from the characteristic polynomial, that the Jordan normal form consists of 0- and -1-blocks, the sum of the sizes of the former is 5, of the latter is 2. From the minimal polynomial we can deduce that the largest 0-block is of size 3, and the largest -1-block has size 1. So there are only two possibilities: the diagonal blocks of the Jordan normal form are:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, [0], \; [0], \; [-1], \; [-1] \qquad \text{or} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; [-1] \; [-1]$$

b) By the characteristic polynomial, the diagonal blocks of the Jordan normal form can only be 1-blocks and one 1×1 0-block. The sum of the sizes of the 1-blocks is 4, and we know by the dimension of the eigenspace that there are exactly 2 1-blocks. So there are two possibilities:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, [1], \; [0] \qquad \text{or} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; [0]$$

5. *Find two non-similar 7×7 matrices which have the same minimal and characteristic polynomials, and their eigenspaces also have the same dimension.*

Solution: We can take the two Jordan-matrices, with diagonal blocks

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, [0], \qquad \text{and} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

6. *Calculate the nth power of the following matrices, using the diagonal or Jordan normal form.*

$$A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$$

Solution: $|A - xI| = x^2 - x - 2 = (x-2)(x+1)$, so the eigenvalues of A are 2 and -1. The eigenvectors corresponding to 2 are the scalar multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the eigenvectors corresponding to -1 are the scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So with the transition matrix $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, we have $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D$. This implies that $A^n = (PDP^{-1})^n = PD^nP^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2^{n+1} - (-1)^n & -2^{n+1} + 2 \cdot (-1)^n \\ 2^n - (-1)^n & -2^n + 2 \cdot (-1)^n \end{bmatrix}$

$|B - xI| = x^2 - 4x + 4 = (x-2)^2$, so the only eigenvalue of B is 2. Since the rank of $B - 2I$ is 1, the dimnesion of the eigenspace corresponding to 2 is $2 - 1 = 1$, thus the Jordan normal form of B consists of one single Jordan block: $P^{-1}BP = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ with an appropriate transition matrix P. The column vectors of P (that is, the elements of the new basis), $\mathbf{b}_1$ and $\mathbf{b}_2$ must satisfy the equations: $B\mathbf{b}_1 = 2\mathbf{b}_1$ and $B\mathbf{b}_2 = \mathbf{b}_1 + 2\mathbf{b}_2$. The first shows that $\mathbf{b}_1$ must be an eigenvector for 2, and this we can find as a solution of the equation $(B - 2I)\mathbf{x} = \mathbf{0}$, say, $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. If we substitute this into the second then $\mathbf{b}_2$ will be a solution of $B\mathbf{x} = \mathbf{b}_1 + 2\mathbf{x}$, that is, $(B - 2I)\mathbf{x} = \mathbf{b}_1$, and solving this, we get $\mathbf{b}_2 = \begin{bmatrix} 1 + 2t \\ t \end{bmatrix}$, say, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This gives $P = [\mathbf{b}_1 | \mathbf{b}_2] = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, and $P^{-1}BP = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = J$, so $B^n = PJ^nP^{-1}$. If we denote by N the nilpotent matrix $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the nth power of the Jordan block is $J^n = (2I + N)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} I N^k = \binom{n}{0} 2^n I + \binom{n}{1} 2^{n-1} N = \begin{bmatrix} 2^n & n \cdot 2^{n-1} \\ 0 & 2^n \end{bmatrix}$, since $N^2 = 0$. From this we get $B^n = \begin{bmatrix} (n+1) \cdot 2^n & -n \cdot 2^{n+1} \\ n \cdot 2^{n-1} & (1-n) \cdot 2^n \end{bmatrix}$.

7. *Prove that every $n \times n$ complex matrix is similar to its transposed matrix. (Use the Jordan normal form.)*

Solution: First we show that the statement is true form Jordan blocks. If $J_{n\times n}$ is a 0-block, then the transformation $\mathbf{x} \mapsto J\mathbf{x}$ acts on the basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ as $\mathbf{e}_n \mapsto \mathbf{e}_{n-1} \mapsto \cdots \mapsto \mathbf{e}_1 \mapsto \mathbf{0}$. If in a basis $\mathcal{B} = \mathbf{b}_1, \dots, \mathbf{b}_n$ the vectors are the same, only in reverse order, then $\mathbf{b}_1 \mapsto \cdots \mapsto \mathbf{b}_{n-1} \mapsto \mathbf{b}_n \mapsto \mathbf{0}$, so J in this basis will be J^T.

For a Jordan block J corresponding to λ, the 0-block $J - \lambda I$ is similar to $(J - \lambda I)^T$ by the previous paragraph. If P is the transition matrix, then $P^{-1}JP - \lambda I = P^{-1}(J - \lambda I)P = (J - \lambda I)^T = J^T - \lambda I$, so $P^{-1}JP = J^T$, that is, J is similar to J^T.

If the corresponding blocks of two block diagonal matrix are similar, then the two block matrices are also similar (the transition matrix is the block diagonal matrix whose diagonal blocks are the transition matrices given for the similarity of the blocks). So every Jordan matrix is similar to its transposed matrix.

Finally, if two matrices are similar: $B = P^{-1}AP$, then their transposed matrices are also similar: $B^T = P^T A^T (P^T)^{-1}$. So for a matrix A and its Jordan normal form J, we have $A \sim J \sim J^T \sim A^T$, consequently A itself is similar to its transposed matrix.

8. *Is there a matrix $I \neq A \in \mathbb{Q}^{n\times n}$ such that a) $A^3 = I$; b) $A^5 = I$? And in $\mathbb{Q}^{2\times 2}$?*

Solution: a) The 3×3 matrix, acting on the standard basis as $\mathbf{e}_1 \mapsto \mathbf{e}_2 \mapsto \mathbf{e}_3 \mapsto \mathbf{e}_1$ clearly satisfies the given conditions. If we want a 2×2 matrix, its minimal polynomial is a divisor of $x^3 - 1 = (x-1)(x^2 + x + 1)$ (since A is a 'root' of this polynomial), and the degree of the minimal polynomial is at most 2, and it is not $x - 1$, since $A \neq I$, so it can only be $x^2 + x + 1$, and then $k_A(x) = m_A(x) = x^2 + x + 1$. It is easy to construct such a matrix, for instance, $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$.

b) We can construct a 5×5 matrix similarly to the first example of part a): it can be the matrix acting on the basis elements as $\mathbf{e}_1 \mapsto \mathbf{e}_2 \mapsto \cdots \mapsto \mathbf{e}_5 \mapsto \mathbf{e}_1$. If there were a 2×2 matrix A satisfying the given condition, then $m_A(x)$ would be a divisor of $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$, and the degree of $m_A(x)$ would be at most 2. But the second polynomial has no proper factors over $\mathbb{Q}$ (over $\mathbb{R}$ we can factor it as $(x^2 - \frac{-1+\sqrt{5}}{2}x + 1)(x^2 - \frac{-1-\sqrt{5}}{2}x + 1)$), so $m_A(x)$ could only be $x - 1$, but then A would be the identity matrix I.