1. *Write the vector* $\mathbf{b}$ *as the sum of a vector which is orthogonal to* $\mathbf{a}$ *and a vector which is parallel to* $\mathbf{a}$ *if*

a) $\mathbf{a} = (1, -2, 0, 1)$, $\mathbf{b} = (3, 1, 1, 1)$;

b) $\mathbf{a} = (1 + i, 1 - i)$, $\mathbf{b} = (i, 3 - i)$.

Solution: a) The component parallel with $\mathbf{a}$ is the orthogonal projection of $\mathbf{b}$ on $\mathbf{a}$, which is $\mathbf{b}' = \frac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}|^2}\mathbf{a} = \frac{2}{6}\mathbf{a} = \left(\frac{1}{3}, -\frac{2}{3}, 0, \frac{1}{3}\right)$, and the component orthogonal to $\mathbf{a}$ is $\mathbf{b} - \mathbf{b}' = \left(\frac{8}{3}, \frac{5}{3}, 1, \frac{2}{3}\right)$.

b) For the orthogonal projection $\mathbf{b}' = \frac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}|^2}\mathbf{a}$, we need $\langle \mathbf{a}, \mathbf{b}\rangle = (1 - i)i + (1 + i)(3 - i) = 5 + 3i$ and $|\mathbf{a}|^2 = |1 + i|^2 + |1 - i|^2 = 1^2 + 1^2 + 1^2 + 1^2 = 4$. So the component parallel with $\mathbf{a}$ is $\mathbf{b}' = \frac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}|^2}\mathbf{a} = \frac{5+3i}{4}\mathbf{a} = \left(\frac{1}{2} + 2i,\ 2 - \frac{1}{2}i\right)$, and the component orthogonal to $\mathbf{a}$ is $\mathbf{b} - \mathbf{b}' = \left(-\frac{1}{2} - i,\ 1 - \frac{1}{2}i\right)$.

2. *Suppose* $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathbb{R}^n$ *are orthogonal vectors and neither of them is* $\mathbf{0}$. *Let* $W = \operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$, *and for a vector* $\mathbf{v} \in \mathbb{R}^n$ *define* $\mathbf{v}' = \sum_{i=1}^{k} \frac{\mathbf{b}_i^T\mathbf{v}}{|\mathbf{b}_i|^2}\mathbf{b}_i$. *Prove that*

a) $\mathbf{v} - \mathbf{v}' \perp \mathbf{w}$ *for every* $\mathbf{w} \in W$;

b) if $\mathbf{v} \notin W$, *then* $\{\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{v} - \mathbf{v}'\}$ *is an orthogonal basis of* $\operatorname{span}(W \cup \{\mathbf{v}\})$;

c) $\mathbf{v}'$ *is the element of* W *closest to* $\mathbf{v}$ *(that is,* $|\mathbf{v} - \mathbf{v}'| = \min\{|\mathbf{v} - \mathbf{w}| \mid \mathbf{w} \in W\}$*).*

Solution: a) Since the $\mathbf{b}_i$ are pairwise orthogonal,

$$\langle \mathbf{b}_j, \mathbf{v}'\rangle = \sum_{i=1}^{k} \langle \mathbf{b}_j, \frac{\mathbf{b}_i^T\mathbf{v}}{|\mathbf{b}_i|^2}\mathbf{b}_i\rangle = \sum_{i=1}^{k} \frac{\mathbf{b}_i^T\mathbf{v}}{|\mathbf{b}_i|^2}\langle \mathbf{b}_j, \mathbf{b}_i\rangle = \mathbf{b}_j^T\mathbf{v} = \langle \mathbf{b}_j, \mathbf{v}\rangle,$$

so $\langle \mathbf{b}_j,\ \mathbf{v} - \mathbf{v}'\rangle = 0$ for any $j \in \{1, \dots, k\}$. But then for any $\mathbf{w} = \sum_{j=1}^{k} x_j\mathbf{b}_j \in W$ we have $\langle \mathbf{w},\ \mathbf{v} - \mathbf{v}'\rangle = \sum_{j=1}^{k} \overline{x_j}\langle \mathbf{b}_j,\ \mathbf{v} - \mathbf{v}'\rangle = 0$.

b) First of all, $\operatorname{span}(W \cup \{\mathbf{v}\}) = \operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{v}\} = \operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{v} - \mathbf{v}'\}$, since $\mathbf{v}' \in \operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$. Furthermore, the set $\{\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{v} - \mathbf{v}'\}$ is linearly independent: if a nontrivial linear combination of the vectors is 0, and the last coefficient is nonzero, then $\mathbf{v}$ can be expressed from the others, contradicting the assumption that $v \notin W$; so it would be a combination of the vectors $\mathbf{b}_i$, but those are linearly independent. Finally, this set is orthogonal by part a).

c) For any $\mathbf{w} \in W$, we have $|\mathbf{v} - \mathbf{w}|^2 = |(\mathbf{v} - \mathbf{v}') + (\mathbf{v}' - \mathbf{w})|^2 =$ $= \langle(\mathbf{v}-\mathbf{v}')+(\mathbf{v}'-\mathbf{w}),\ (\mathbf{v}-\mathbf{v}')+(\mathbf{v}'-\mathbf{w})\rangle = \langle(\mathbf{v}-\mathbf{v}'),\ (\mathbf{v}-\mathbf{v}')\rangle+\langle(\mathbf{v}'-\mathbf{w}),\ (\mathbf{v}'-\mathbf{w})\rangle$, since $\mathbf{v}-\mathbf{v}'$ is orthogonal to $\mathbf{v}' - \mathbf{w}$ by part a). Thus $|\mathbf{v}-\mathbf{w}|^2 = \langle(\mathbf{v}-\mathbf{v}'),\ (\mathbf{v}-\mathbf{v}')\rangle+\langle(\mathbf{v}'-\mathbf{w}),\ (\mathbf{v}'-\mathbf{w})\rangle = |\mathbf{v} - \mathbf{v}'|^2 + |\mathbf{v}' - \mathbf{w}|^2 \geq |\mathbf{v} - \mathbf{v}'|^2$, and the equation holds if and only if $\mathbf{w} = \mathbf{v}'$.

3. *Use problem 2.b) to find and orthogonal and then an orthonormal basis in the subspace of* $\mathbb{R}^4$ *spanned by* $(1, 2, -1, 0)$, $(2, 1, 0, 1)$ *and* $(1, -1, 1, -1)$.

Solution: Let the three given vectors be called $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. We replace them one by one using 2.b) so that the spanned subspaces of the first k vectors ($k = 1, 2, 3$) remain the same. The first vector can remain $\mathbf{c}_1 = \mathbf{b}_1 = (1, 2, -1, 0)$. Then $\mathbf{b}_2 - \mathbf{b}_2' = \mathbf{b}_2 - \frac{\langle \mathbf{c}_1, \mathbf{b}_2\rangle}{|\mathbf{c}_1|^2}\mathbf{c}_1 = (2, 1, 0, 1) - \frac{4}{6}(1, 2, -1, 0) = (\frac{4}{3}, -\frac{1}{3}, \frac{2}{3}, 1)$, but we can choose a nonzero scalar multiple instead: $\mathbf{c}_2 = (4, -1, 2, 3)$. Then $\mathbf{b}_3 - \frac{\langle \mathbf{c}_1, \mathbf{b}_3\rangle}{|\mathbf{c}_1|^2}\mathbf{c}_1 - \frac{\langle \mathbf{c}_2, \mathbf{b}_3\rangle}{|\mathbf{c}_2|^2}\mathbf{c}_2 = (1, -1, 1, -1) + \frac{2}{6}(1, 2, -1, 0) - \frac{4}{30}(4, -1, 2, 3) = (\frac{4}{5}, -\frac{1}{5}, \frac{2}{5}, -\frac{7}{5})$, so $\mathbf{c}_3$ can be a scalar multiple of this: $\mathbf{c}_3 = (4, -1, 2, -7)$. Thus $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\} = \{(1, 2, -1, 0),\ ((4, -1, 2, 3),\ (4, -1, 2, -7)\}$ is an orthogonal basis of the given subspace, and $\left\{\frac{\mathbf{c}_1}{|\mathbf{c}_1|}, \frac{\mathbf{c}_2}{|\mathbf{c}_2|}, \frac{\mathbf{c}_3}{|\mathbf{c}_3|}\right\} = \left\{\frac{1}{\sqrt{6}}(1, 2, -1, 0),\ \frac{1}{\sqrt{30}}(4, -1, 2, 3),\ \frac{1}{\sqrt{70}}(4, -1, 2, -7)\right\}$ is an orthonormal basis.

4. *Prove that the subset* $\{(x_1,x_2,x_3,x_4,x_5)\,|\,x_1+x_2=x_4+x_5\}$ *is a hyperplane in* $\mathbb{R}^5$*, and determine its normal vector. Calculate the reflection of* $(1,0,0,0,0)$ *to this hyperplane.*

Solution: The subset consists of those vectors $\mathbf{x}$ whose scalar product with $(1,1,0,-1,-1)$ is 0, so it is the hyperplane with normal vector $\mathbf{a}=(1,1,0,-1,-1)$. We can obtain the reflection of a vector $\mathbf{v}$ on the hyperplane as $\mathbf{v}-\frac{2}{|\mathbf{a}|^2}\langle\mathbf{a},\mathbf{v}\rangle\mathbf{a}$, so the reflection of $\mathbf{v}=(1,0,0,0,0)$ is $(1,0,0,0,0)-\frac{2}{4}(1,1,0,-1,-1)=(\frac{1}{2},-\frac{1}{2},0,\frac{1}{2},\frac{1}{2})$.

5. *Give the standard matrix of the orthogonal projection and of the reflection on the hyperplane* $x+y-z=0$ *in* $\mathbb{R}^3$.

Solution: The normal vector of the plane is $(1,1,-1)$, so the standard matrix of the orthogonal projection is $I-\frac{1}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*$, where $\mathbf{a}$ is the normal vector written as a column vector:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{1}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix}\begin{bmatrix}1&1&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{1}{3}\begin{bmatrix}1&1&-1\\1&1&-1\\-1&-1&1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2&-1&1\\-1&2&1\\1&1&2\end{bmatrix}.$$

The standard matrix of the reflection is $I-\frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*$:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{2}{3}\begin{bmatrix}1&1&-1\\1&1&-1\\-1&-1&1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1&-2&2\\-2&1&2\\2&2&1\end{bmatrix}$$

6. *Find the standard matrix of a reflection which maps the vector* $(1,2,-2)$ *to* $(3,0,0)$*. (Hint: It is the reflection on the bisector plane of the line segment connecting the endpoints of the two vectors.)*

Solution: The bisector plane has a normal vector $(3,0,0)-(1,2,-2)=(2,-2,2)$, or its scalar multiple, $(1,-1,1)$, and the plane contains the origin, since $|(1,2,-2)|=3=|(3,0,0)|$. So it is a hyperplane $H(\mathbf{a})$ with $\mathbf{a}=\begin{bmatrix}1&-1&1\end{bmatrix}^T$. Thus the standard matrix of the reflection is

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}1&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{2}{3}\begin{bmatrix}1&-1&1\\-1&1&-1\\1&-1&1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1&2&-2\\2&1&2\\-2&2&1\end{bmatrix}.$$

One can easily check that this transformation maps $(1,2,-2)$ to $(3,0,0)$.

7. *Which of the following matrices are self-adjoint, unitary or normal? Which of the self-adjoint matrices are positive semidefinite or positive definite?*

$$A=\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}\qquad B=\begin{bmatrix}0&2&-1\\-2&0&3\\1&-3&0\end{bmatrix}\qquad C=\begin{bmatrix}i&i\\i&-i\end{bmatrix}\qquad D=\begin{bmatrix}2&1\\3&2\end{bmatrix}$$

$$E=\begin{bmatrix}1&-1\\-1&3\end{bmatrix}\qquad F=\begin{bmatrix}-1&2+i\\2-i&-5\end{bmatrix}\qquad G=\begin{bmatrix}1/3&-2/3&-2/3\\2/3&2/3&-1/3\\2/3&-1/3&2/3\end{bmatrix}\qquad H=\begin{bmatrix}1&i\\1+i&0\end{bmatrix}$$

Solution: A, E, F are self-adjoint, G is unitary. $B^*=-B$, so $B^*B=BB^*$, that is, B is normal. $C=i\begin{bmatrix}1&1\\1&-1\end{bmatrix}$ is a scalar multiple of a self-adjoint (so also normal) matrix, thus C is normal. (If M is normal, then $(cM)^*(cM)=|c|^2M^*M=|c|^2MM^*=(cM)(cM)^*$.) D and H are not even normal, because $D^*D\neq DD^*$ and $H^*H\neq HH^*$.

We can determine the definiteness A and E by simultaneous row and column operations:

$$A=\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}\mapsto\begin{bmatrix}1&1&1\\0&-1&0\\0&0&0\end{bmatrix}\mapsto\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix},\qquad\text{so } A \text{ is indefinite.}$$

$$E=\begin{bmatrix}1&-1\\-1&3\end{bmatrix}\mapsto\begin{bmatrix}1&-1\\0&2\end{bmatrix}\mapsto\begin{bmatrix}1&0\\0&2\end{bmatrix},\qquad\text{so } E \text{ is positive definite.}$$

We check the definiteness of the complex self-adjoint matrix F by calculating its eigenvalues: $|F-xI|=x^2-6x+5-5=x(x-6)$, so the eigenvalues are 6 and 0, thus F is positive semidefinite.

8. *Give an example of a transformation $\mathbb{C}^n \to \mathbb{C}^n$ such that the absolute value of every eigenvalue is 1 but the transformation is not unitary.*

Solution: An example can be the Jordan matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, or a matrix (with eigenvalues of absolute value 1), which is diagonalizable but not with a unitary matrix: $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$.

9. *Give the reduced (and the full) singular value decomposition of the following matrices.*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad B = [\,4 \quad -3\,] \qquad C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \qquad E = \begin{bmatrix} 2 & -11 \\ 10 & -5 \end{bmatrix}$$

Solution: A is a symmetric matrix with eigenvalues 2 and 0, so A is also positive semidefinite, thus its spectral decomposition with a unitary transition matrix is a full SVD. The normal eigenvectors of A are: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for 2, and $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for 0. Hence with the matrices $M = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and $\Sigma' = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ we have $A = M\Sigma' M^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. The reduced SVD (with $r = 1$) is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}[\,2\,]\frac{1}{\sqrt{2}}[\,1 \quad 1\,]$.

$B^T B = \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix}$, its eigenvalues are 25 and 0, the only singular value of B is 5, so $\Sigma = [\,5\,]$. For $\lambda_1 = 25$, a normal eigenvector of $B^T B$ is $\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$, so

$$V = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}, \quad U = BV\Sigma^{-1} = [\,-1\,], \quad B = U\Sigma V^T = [\,-1\,][\,5\,][\,-4/5 \quad 3/5\,].$$

We can complete V to an orthogonal matrix by adding as a new column a unit vector perpendicular to the first column: $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$. So $M = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$. U is already an orthogonal matrix, thus $M' = U = [\,-1\,]$. Finally we get Σ' by completing Σ with 0's to a 1×2 matrix: $\Sigma' = [\,5 \quad 0\,]$. The full SVD of B is

$$B = M'\Sigma' M^T = [\,-1\,][\,5 \quad 0\,]\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}.$$

$$C^T C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalues of $C^T C$ are 2 and 1 with orthonormal eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, respectively. The singular values of C are $\sqrt{2}$ and 1, so $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$.

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; U = CV\Sigma^{-1} = \begin{bmatrix} 0 & -1 \\ \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \text{ so } C = U\Sigma V^T = \begin{bmatrix} 0 & -1 \\ \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the reduced SVD of C. To get the full SVD, we complete U to an orthogonal matrix with the column $\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$ to get M' and keep the orthogonal matrix V as M, finally we make Σ a 3×2 matrix by adding extra 0 elements. So the full SVD of C is:

$$C = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$D^TD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -10 \\ 0 & -10 & 20 \end{bmatrix}$, its eigenvalues are 25, 1 and 0, the singular values of D are 5 and 1, and $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$. The normal eigenvectors of D^TD form an orthonormal basis in $\mathbb{R}^3$:

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \text{ for } 25, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ for } 1, \ \mathbf{b}_3 = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \text{ for } 0$$

So we get immediately V for the reduced and M for the full SVD.

$$V = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \end{bmatrix}, \quad U = DV\Sigma^{-1} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{5}} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \end{bmatrix}$$

For the full SVD we already have the columns of M and we can complete U similary to an orthogonal matrix M':

$$D = M'\Sigma' M^T = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

$E^TE = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}$, its eigenvalues are 200 and 50, and its normal eigenvectors: $\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ and $\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$, and the singular values of E are $10\sqrt{2}$ and $5\sqrt{2}$. So

$$V = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}, \quad U = EV\Sigma^{-1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$E = U\Sigma V^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix},$$

and this is also the full SVD.

10. *Use the reduced SVD form of the matrices of problem 9 to*

a) find the pseudoinverse of A, and with that the best approximate solution of the inconsistent equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$*;*

b) find the matrix of rank 1 *closest to the matrix D.*

Solution: a) $A^+ = U\Sigma^{-1}(U')^* = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} [1/2] [1/\sqrt{2} \quad 1/\sqrt{2}] = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$, and from this the best approximate solution is $\mathbf{x} = A^+ \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 3/4 \end{bmatrix}$. (If we substitute this into the equation, we get $A\mathbf{x} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$ instead of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, but this is the closest that we can get.)

b) $D^{(1)} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} [5] \begin{bmatrix} 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix}$