1. *Determine the standard matrix of the linear transformation* $f(x,y,z) = (x+z, y, x+y+z)$*, the basis of* $\operatorname{Ker} f$ *and* $\operatorname{Im} f$*, and the matrix of* f *in the basis* $\mathcal{B} = \{(1,0,0),(0,1,1),(0,2,1)\}$

Solution: Let A be the standard matrix, $B = [f]_{\mathcal{B}}$, and P the transition matrix. Then

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ y \\ x+y+z \end{bmatrix} \quad \forall x,y,z \Rightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

With Gaussian elimination:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the row-echelon form of A, with leading ones in the first and second column, so the first and second column of A form a basis of $\operatorname{Im} A$: $\{(1,0,1),(0,1,1)\}$, and the solution of

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \text{ is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{ the basis of } \operatorname{Ker} f \text{ is } \{(-1,0,1)\}.$$

Finally,

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad [P|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \mapsto$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow$$

$$B = P^{-1}AP = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -1 \end{bmatrix}.$$

2. *Find the diagonal form of the matrix* $A = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}$*, together with the transition matrix. Calculate* A^n *for any natural number* n*.*

Solution: $k_A(x) = |A - xI| = \begin{vmatrix} -1-x & 1 \\ -3 & 3-x \end{vmatrix} = x^2 - 2x = x(x-2)$, so the eigenvalues of A are 2 and 0. The eigenvectors for 2 are the solutions of $\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}\mathbf{v} = \mathbf{0}$, that is, $\mathbf{v} = \begin{bmatrix} \frac{1}{3}t \\ t \end{bmatrix} = \frac{t}{3} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, for 0 the solutions of $\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\mathbf{v} = \mathbf{0}$, that is, $\mathbf{v} = \begin{bmatrix} t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis of eigenvectors. The transition matrix is $P = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$, and $P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

$$[P|I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \Rightarrow$$

$$P^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 0^n \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2^{n-1} & 2^{n-1} \\ -3 \cdot 2^{n-1} & 3 \cdot 2^{n-1} \end{bmatrix}.$$

3. *What can be the Jordan normal form of a matrix* A *whose characteristic polynomial is* $k_A(x) = (x+1)^4x^2$ *and whose minimal polynomial is* $m_A(x) = (x+1)^2x^2$*. Give the dimension of the eigenspaces is each case.*

Solution: From the characteristic polynomial: the eigenvalue -1 has multiplicity 4, and 0 has multiplicity 2. From the minimal polynomial: the largest -1-block is of size 2, the largest 0-block is of size 2. So there can only be one 0-block, but the sizes of the -1-blocks can be $2,2$ or $2,1,1$:

$$J_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad J_2 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\dim V_{-1} = 2$ in the first case and 3 in the second (the number of -1-blocks), and $\dim V_0 = 2$ in both cases (the number of 0-blocks).

4. *Which of the following matrices are self-adjoint, normal or unitary? Determine the definiteness of those which are self-adjoint.*

$$A = \begin{bmatrix} i & 1 \\ -1 & 1+i \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & -1 & 0 \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix}$$

Solution: A is not even normal:

$$A^*A = \begin{bmatrix} 2 & -1-2i \\ -1+2i & 3 \end{bmatrix} \quad \text{and} \quad AA^* = \begin{bmatrix} 2 & 1-2i \\ 1+2i & 3 \end{bmatrix} \neq A^*A.$$

B is self-adjoint because it is real symmetric: $B^* = B^T = B$, so B is also normal.
C is unitary because its columns form an orthonormal basis in $\mathbb{R}^3$: they have length 1, and their scalar products are 0. So C is also normal.
By simultaneous row-column operations:

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \overset{row}{\mapsto} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \overset{col.}{\mapsto} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \overset{row}{\mapsto}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{col.}{\mapsto} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The diagonal matrix contains positive numbers and zeros in the diagonal, so B is positive semidefinite.
(Or one can calculate the eigenvalues: $k_B(x) = -x^3 + 5x^2 - 6 = x(x-2)(x-3)$, so the eigenvalues are $0, 2, 3$ all nonnegative, thus B is positive semidefinite.)

5. *Determine the reduced SVD of* $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$, *and the best approximating matrix of rank* 1.

Solution: $A^TA = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $k_{A^TA} = x^2 - 6x + 8 = (x-2)(x-4)$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\sigma_1 = 2$, $\sigma_2 = \sqrt{2}$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$.

The eigenvectors of A^TA for $\lambda_1 = 4$ are the solutions of $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$, which are the scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so a unit eigenvector is $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

The eigenvectors of A^TA for $\lambda_1 = 2$ are the solutions of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$, which are the scalar multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so a unit eigenvector is $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \qquad U = AV\Sigma^{-1} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix} \quad \Rightarrow$$

$$A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

The rank 1 approximation is

$$A^{(1)} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \cdot [2] \cdot [1/\sqrt{2} \quad 1/\sqrt{2}] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

6. *State the following definitions and theorems*

a) rank of a matrix;
b) the theorem of polynomial interpolation;
c) standard scalar product in $\mathbb{C}$;
d) theorem about the connection between diagonalizability and the minimal polynomial of a matrix;
e) singular values.

Solution: a) The rank of a matrix $A \in K^{m\times n}$ is the dimension of the column space of the matrix, that is, the maximum of the number of columns in A which are linearly independent.

b) Let K be a field, and suppose that $a_0, a_1, \ldots, a_n \in K$ are different, $b_0, b_1, \ldots, b_n \in K$. Then there exists a unique polynomial $f(x) \in K[x]$ of degree at most n such that $f(a_i) = b_i$ for $i = 0, 1, \ldots, n$.

c) For the column vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, the scalar product of $\mathbf{x}$ and $\mathbf{y}$ is $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* \mathbf{y} = \sum_{j=1}^n \overline{x}_j y_j$.

d) Suppose that $A \in K^{n\times n}$, and $k_A(x)$ can be written as a product of linear polynomials. Then A is diagonalizable if and only if every root of $m_A(x)$ has multiplicity 1.

e) For $A \in \mathbb{R}^{m\times n}$, $\text{rank}(A) = r$, the singular values of A are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ if the positive eigenvalues of $A^T A$ are $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2$.