Eigenvalues, eigenvectors, diagonalization

Def. $\mathbf{v} \in V_K$ is an **eigenvector** of the linear transformation $f : V \to V$ if $\mathbf{v} \neq \mathbf{0}$, and there is a scalar $\lambda \in K$ such that $f(\mathbf{v}) = \lambda\mathbf{v}$, that is, $f(\mathbf{v})$ is parallel to $\mathbf{v}$ (including the case when $f(\mathbf{v}) = \mathbf{0}$).
Here λ is the **eigenvalue** corresponding to $\mathbf{v}$.
The **spectrum** of f is the set of eigenvalues of f.
The **eigenspace** corresponding to the eigenvalue λ is $V_\lambda = \{\, \mathbf{v} \in V \mid f(\mathbf{v}) = \lambda\mathbf{v} \,\} \leq V$, which consists of $\mathbf{0}$ and the eigenvectors for λ.

Example: The eigenvectors of an orthogonal projection onto a plane containing the origin are the nonzero vectors of the plane (with eigenvalue 1), and the nonzero vectors orthogonal to the plane (with eigenvalue 0).
In other words, the plane itself is the eigenspace for 1, and the line through the origin which is perpendicular to the plane is the eigenspace for 0.

Def. The **eigenvectors, eigenvalues** and the **spectrum** of a matrix A are those of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Diagonalization (spectral decomposition)

$A \in K^{n\times n}$, $f : K^n \to K^n$, $f : \mathbf{x} \mapsto A\mathbf{x}$. If $\exists$ a basis $\mathcal{B} = \{\, \mathbf{b}_1, \dots, \mathbf{b}_n \,\}$ consisting of eigenvectors of f with eigenvalues $\lambda_1, \dots, \lambda_n$ then

$$[f]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} = D$$

is a diagonal matrix.
With the transition matrix $P = [\mathbf{b}_1 \dots \mathbf{b}_n]$ we have $D = P^{-1}AP$, that is, $A = PDP^{-1}$. The latter is the **spectral decomposition** of A.

Def. $A \in K^{n\times n}$ is **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal (in other words, A is similar to a diagonal matrix), i.e. $\exists$ a basis in K^n consisting of eigenvectors of A.

Powers of diagonalizable matrices

If $A = PDP^{-1}$, then $A^k = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = PD^kP^{-1}$, and we obtain the kth power of a diagonal matrix simply by taking the kth powers of the diagonal elements.

Calculating eigenvalues and eigenvectors

$$\exists \mathbf{v} \neq \mathbf{0} : A\mathbf{v} = \lambda\mathbf{v} \;\Leftrightarrow$$
$$\exists \mathbf{v} \neq \mathbf{0} : (A - \lambda I)\mathbf{v} = \mathbf{0} \;\Leftrightarrow$$
$$|A - \lambda I| = 0$$

Characteristic polynomial

Def. The **characteristic polynomial** of the matrix A is

$$k_A(x) = |A - xI| = \begin{vmatrix} a_{11}-x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-x & \dots & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn}-x \end{vmatrix}$$

Then the eigenvalues of A are exactly the roots of the characteristic polynomial $k_A(x)$.

Exercises: Which of the following matrices are diagonalizable over $\mathbb{R}$ or $\mathbb{C}$?

$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$|A - xI| = \begin{vmatrix} 1-x & 2 \\ 0 & 2-x \end{vmatrix}$

$= (x-1)(x-2)$

eigenvalues: $\lambda = 1, 2$

$\exists$ eigenvector for each,,

they are indep. $\Rightarrow$

they form a basis $\Rightarrow$

A is diag.-able

$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$|B - xI| = x^2 + 1$

no real root $\Rightarrow$

B is not diag.-able over $\mathbb{R}$

(but diag.-able over $\mathbb{C}$)

$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$|C - xI| = (x-1)^2$

eigenvalue: $\lambda = 1$

$(C - 1 \cdot I)\mathbf{v} = \mathbf{0}$, $\mathbf{v} =?$

$\begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} t \\ 0 \end{bmatrix}$

$\nexists$ two indep. eigenvectors $\Rightarrow$

C is not diagonalizable

(neither over $\mathbb{R}$ nor over $\mathbb{C}$)

Proposition. *If $k_A(x)$ can be written as the product of linear polynomials, i.e. $k_A(x) = (-1)^n(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then*

$$\lambda_1 + \ldots + \lambda_n = \operatorname{tr} A = a_{11} + a_{22} + \ldots + a_{nn}$$

($\operatorname{tr} A$ is called the trace of A), and

$$\lambda_1 \cdots \lambda_n = |A|.$$

Proof. In the summands of the determinant $|A - xI|$, x^{n-1} can only appear when we multiply the elements of the diagonal, and in that product the coefficient of x^{n-1} is $(-1)^{n-1}(a_{11} + \ldots + a_{nn})$, which is the same as the coefficient of x^{n-1} in the factorization of $k_A(x)$, and that is $(-1)^n(-\lambda_1 - \lambda_2 - \ldots - \lambda_n) = (-1)^{n-1}(\lambda_1 + \ldots + \lambda_n)$.

The constant term of the polynomial $k_A(x)$ is $k_A(0) = |A - 0I| = |A|$, while the constant term in the factored form is $(-1)^n(-\lambda_1)\cdots(-\lambda_n) = \prod \lambda_i$.

Def. For $A \in K^{n\times n}$ and $p(x) = c_m x^m + \ldots + c_1 x + c_0 \in K[x]$, we define $p(A) := c_m A^m + \ldots + c_1 A + c_0 I$.

Cayley–Hamilton theorem. $k_A(A) = 0$.

Example: For $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $k_A(x) = (x-1)(x-2)$, and

$$(A - I)(A - 2I) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$