Euclidean spaces and their transformations

Scalar product (dot product) in $\mathbb{R}^3$

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cdot\cos\alpha$, where α is the angle of the two vectors.
With coordinates: if $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ then $\mathbf{ab} = a_1b_1 + a_2b_2 + a_3b_3$.

Exercise Consider the unit cube $0 \le x, y, z \le 1$, and let $\mathbf{a} = (1, 0, 1)$ and $\mathbf{b} = (0, 1, 1)$ be the diagonal vectors of two faces of the cube starting from the origin. What is the angle of $\mathbf{a}$ and $\mathbf{b}$?
Solution: $\mathbf{ab} = 0+0+1 = 1$, $|\mathbf{a}| = |\mathbf{b}| = \sqrt{2} \Rightarrow \cos\alpha = \frac{1}{2} \Rightarrow \alpha = 60°$. Indeed, the corners $(0,0,0)$, $(1,0,1)$ and $(0,1,1)$ form an equilateral triangle, since the third side is also the diagonal of a face of the cube.

Properties:

- $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (perpendicular) $\Leftrightarrow \mathbf{ab} = 0$. Notation: $\mathbf{a}\perp\mathbf{b}$.
- $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$, so $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$
- Projection of a vector $\mathbf{x}$ onto a vector $\mathbf{a} \ne \mathbf{0}$: $\mathbf{x}' = \dfrac{\mathbf{a}\cdot\mathbf{x}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \dfrac{\mathbf{ax}}{|\mathbf{a}|^2}\mathbf{a}$,
  since $\frac{\mathbf{ax}}{|\mathbf{a}|^2}\mathbf{a} = \frac{|\mathbf{a}|\cdot|\mathbf{x}|\cos\alpha}{|\mathbf{a}|^2}\mathbf{a} = |\mathbf{x}|\cos\alpha\cdot\frac{\mathbf{a}}{|\mathbf{a}|}$, where $|\mathbf{x}|\cos\alpha$ is the length of the projection (with + or − sign) and $\frac{\mathbf{a}}{|\mathbf{a}|}$ is the unit vector pointing in the same direction as $\mathbf{a}$.

Scalar product in $\mathbb{R}^n$ and in $\mathbb{C}^n$

We consider the elements of $\mathbb{R}^n$ and $\mathbb{C}^n$ as column vectors.

Def.: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the (standard) scalar product of $\mathbf{x}$ and $\mathbf{y}$ is $\langle\mathbf{x},\mathbf{y}\rangle := \mathbf{x}^T\mathbf{y} = \sum_{j=1}^n x_jy_j$ (the 1×1 matrix taken as a scalar).
For $\mathbf{x} \in \mathbb{R}^n$, the vector $\mathbf{x}^T$ is the **transposed vector** of $\mathbf{x}$, which is the row vector $[x_1 \ldots x_n]$.
For $\mathbf{x} \in \mathbb{C}^n$, the vector $\mathbf{x}^* = \overline{\mathbf{x}^T} = [\overline{x_1}, \ldots, \overline{x_n}]$ is the **adjoint vector** of $\mathbf{x}$, which is the same as $\mathbf{x}^T$ if all coordinates of $\mathbf{x}$ are real.
For $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, the (standard) scalar product of $\mathbf{x}$ and $\mathbf{y}$ is $\langle\mathbf{x},\mathbf{y}\rangle := \mathbf{x}^*\mathbf{y} = \sum_{j=1}^n \overline{x_j}y_j$.

$\mathbb{R}^n$ and $\mathbb{C}^n$ with this scalar product are called real or complex **Euclidean spaces**.

We say that $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** to each other in $\mathbb{R}^n$ or $\mathbb{C}^n$ if $\langle\mathbf{x},\mathbf{y}\rangle = 0$.

Def.: For a matrix $A \in \mathbb{C}^{m\times n}$ the **adjoint matrix** $A^* = \overline{A^T}$ is the $n \times m$ matrix whose (i,j) element is $\overline{a_{ji}}$.

Example: For $A = \begin{bmatrix} 1 & 1-i & i \\ 0 & 2+i & 5 \end{bmatrix}$, the adjoint matrix is $A^* = \begin{bmatrix} 1 & 0 \\ 1+i & 2-i \\ -i & 5 \end{bmatrix}$.

Properties of the scalar product

in $\mathbb{R}^n$		in $\mathbb{C}^n$
	$\langle\mathbf{x}+\mathbf{x}',\mathbf{y}\rangle = \langle\mathbf{x},\mathbf{y}\rangle + \langle\mathbf{x}',\mathbf{y}\rangle$	
	$\langle\mathbf{x},\mathbf{y}+\mathbf{y}'\rangle = \langle\mathbf{x},\mathbf{y}\rangle + \langle\mathbf{x},\mathbf{y}'\rangle$	
	$\langle\mathbf{x},\lambda\mathbf{y}\rangle = \lambda\langle\mathbf{x},\mathbf{y}\rangle$	
$\langle\lambda\mathbf{x},\mathbf{y}\rangle = \lambda\langle\mathbf{x},\mathbf{y}\rangle$		$\langle\lambda\mathbf{x},\mathbf{y}\rangle = \overline{\lambda}\langle\mathbf{x},\mathbf{y}\rangle$
$\langle\mathbf{y},\mathbf{x}\rangle = \langle\mathbf{x},\mathbf{y}\rangle$		$\langle\mathbf{y},\mathbf{x}\rangle = \overline{\langle\mathbf{x},\mathbf{y}\rangle}$
	$\langle\mathbf{x},\mathbf{x}\rangle \ge 0$ real, and > 0 if $\mathbf{x} \ne 0$	
	$\lvert\mathbf{x}\rvert := \sqrt{\langle\mathbf{x},\mathbf{x}\rangle}$	
	$\mathbf{x}\perp\mathbf{y} :\Leftrightarrow \langle\mathbf{x},\mathbf{y}\rangle = 0$	

These properties mean that the scalar product in $\mathbb{R}^n$ is a symmetric bilinear form, the scalar product in $\mathbb{C}^n$ is an Hermitian form, and both are positive definite: $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ if $\mathbf{x} \neq \mathbf{0}$.

It can be proved (see Gram–Schmidt orthogonalization) that any subspace of a Euclidean space has an orthonormal basis, that is, a basis $\mathcal{B} = \{ \mathbf{b}_1, \dots, \mathbf{b}_m \}$ such that $|b_i| = 1$ for every i and $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ for every $i \neq j$.

Orthogonal projection on a vector

Proposition: Let $\mathbf{a} \in K^n$, where $K = \mathbb{R}$ or $K = \mathbb{C}$, and assume that $\mathbf{a} \neq \mathbf{0}$. Consider the map

$$\mathbf{x} \mapsto \mathbf{x}' = \frac{\mathbf{a}^*\mathbf{x}}{\mathbf{a}^*\mathbf{a}}\mathbf{a} = \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{|\mathbf{a}|^2}\mathbf{a}$$

Then $\mathbf{x}' || \mathbf{a}$ and $(\mathbf{x} - \mathbf{x}') \perp \mathbf{a}$, which means that $\mathbf{x}'$ is the **orthogonal projection** of $\mathbf{x}$ to $\mathbf{a}$.

Proof: $\mathbf{x}'$ is a scalar multiple of $\mathbf{a}$, so it is parallel to $\mathbf{a}$.
$\langle \mathbf{a}, \mathbf{x} - \mathbf{x}' \rangle = \langle \mathbf{a}, \mathbf{x} \rangle - \langle \mathbf{a}, \mathbf{x}' \rangle = \langle \mathbf{a}, \mathbf{x} \rangle - \langle \mathbf{a}, \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{|\mathbf{a}|^2}\mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{x} \rangle - \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{|\mathbf{a}|^2}\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{x} \rangle - \langle \mathbf{a}, \mathbf{x} \rangle = 0$

Proposition: The orthogonal projection on a vector $\mathbf{a} \neq 0$ is a linear transformation in $\mathbb{R}^n$ or in $\mathbb{C}^n$, and its matrix is

$$\frac{1}{\mathbf{a}^*\mathbf{a}}\mathbf{a}\mathbf{a}^* = \frac{1}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*.$$

Proof: $\frac{1}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^* \cdot \mathbf{x} = \frac{1}{|\mathbf{a}|^2}\mathbf{a}(\mathbf{a}^*\mathbf{x}) = \frac{1}{|\mathbf{a}|^2}(\mathbf{a}^*\mathbf{x})\mathbf{a} = \mathbf{x}'$, since the multiplication from the right by the 1×1 matrix $\mathbf{a}^*\mathbf{x}$ is the same as the multiplication from the left by $\mathbf{a}^*\mathbf{x}$ as a scalar.

Exercise: Find the matrix of the orthogonal projection onto the vector $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$.

Solution: $A = \frac{1}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^* = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$.

Orthogonal projection and reflection to a hyperplane

For a vector $\mathbf{a} \neq \mathbf{0}$ in K^n (where $K = \mathbb{R}$ or $\mathbb{C}$), the **hyperplane** with normal vector $\mathbf{a}$ is $H(\mathbf{a}) = \{ \mathbf{x} \in K^n \,|\, \langle \mathbf{a}, \mathbf{x} \rangle = 0 \}$: the plane formed by the endpoints of the vectors perpendicular to the vector $\mathbf{a}$. $H(\mathbf{a})$ is an $(n-1)$-dimensional subspace in K^n. For instance, hyperplanes in $\mathbb{R}^2$ are the lines going through the origin, in $\mathbb{R}^3$ the planes going through the origin.

The **orthogonal projection** of $\mathbf{x}$ on $H(\mathbf{a})$ is $\mathbf{x} - \mathbf{x}'$, where $\mathbf{x}'$ is the orthogonal projection of $\mathbf{x}$ on $\mathbf{a}$. So the matrix of this transformation is

$$I - \frac{1}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*.$$

The **reflection** of $\mathbf{x}$ on $H(\mathbf{a})$ is $\mathbf{x} - 2\mathbf{x}'$, where $\mathbf{x}'$ is the orthogonal projection of $\mathbf{x}$ on $\mathbf{a}$. So the matrix of this transformation is

$$I - \frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*.$$

Exercise: Find the standard matrix of the reflection of $\mathbb{R}^3$ to the plane $x+y-2z=0$.
Solution: The normal vector is $(1,1,-2)$, so $\mathbf{a}=[\,1 \quad 1 \quad -2\,]^T$, $|\mathbf{a}|^2=6$, and

$$A = I - \frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^* = I - \frac{1}{3}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}.$$

Unitary, self-adjoint and normal matrices

Properties of the adjoint of a matrix:

$(A+B)^* = A^* + B^*$
$(AB)^* = B^*A^*$
$(cA)^* = \overline{c}A^*$
$(A^*)^* = A$
$A^* = A^T$ if $A \in \mathbb{R}^{m\times n}$

Def.: Let $A \in \mathbb{C}^{n\times n}$.
A is **unitary** if $A^* = A^{-1}$, that is, if $A^*A = AA^* = I$.
A is **self-adjoint** if $A^* = A$.
A is **normal** if $A^*A = AA^*$. Clearly, any unitary or self-adjoint matrix is also normal.

If $A \in \mathbb{R}^{n\times n}$ then unitary is also called **orthogonal** and self-adjoint is also called **symmetric**, since in this case $A^T = A$ means that the matrix is symmetric to its main diagonal.

Proposition: The following are equivalent for $A \in K^{n\times n}$ with $K = \mathbb{R}$ or $\mathbb{C}$:
(i) A is unitary;
(ii) the columns of A form an orthonormal basis in K^n;
(iii) the rows of A form an orthonormal basis in K^n;
(iv) the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps an orthonormal basis to an orthonormal basis.

Exercise: Consider the matrices

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix},\ B = \begin{bmatrix} i & i \\ i & 1 \end{bmatrix},\ C = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix},\ D = \begin{bmatrix} 1 & 2+i \\ 2-i & 3 \end{bmatrix},\ E = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}.$$

Which of them are unitary, self-adjoint or normal?
Solution: A is self-adjoint (real symmetric), D is complex self-adjoint, E is unitary (real orthogonal), so they are all normal. B is not even normal but C is normal: $C^*C = CC^* = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$.

Examples: 1. Every real diagonal matrix is self-adjoint.
2. If $A \in \mathbb{R}^{n\times n}$ is skew-symmetric: $A^T = -A$, then A is normal.
3. If A is the matrix of an orthogonal projection or reflection on a hyperplane then it is self-adjoint: $(\mathbf{a}\mathbf{a}^*)^* = \mathbf{a}\mathbf{a}^*$, so $(I - \frac{1}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*)^* = I - \frac{1}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*$ and $(I - \frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*)^* = I - \frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*$.
4. If A is the matrix of a reflection on a hyperplane, then A is unitary:

$$\left(I - \frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*\right)^*\left(I - \frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*\right) = \left(I - \frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*\right)^2 = I - \frac{4}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^* + \frac{4}{|\mathbf{a}|^4}(\mathbf{a}\mathbf{a}^*)^2,$$

and here $(\mathbf{a}\mathbf{a}^*)^2 = \mathbf{a}(\mathbf{a}^*\mathbf{a})\mathbf{a}^* = |\mathbf{a}|^2\mathbf{a}\mathbf{a}^*$, so $AA^* = A^2 = I$.
Note that the eigenvalues of a projection are 0 and 1, the eigenvalues of a reflection are 1 and -1

Theorem
(1) If $A \in \mathbb{C}^{n\times n}$ is unitary, then $|\lambda| = 1$ for every eigenvalue λ of A.
(2) If $A \in \mathbb{C}^{n\times n}$ is self-adjoint, then every eigenvalue of A is real.

Proof: Suppose that $\mathbf{v}$ is an eigenvector with eigenvalue λ: $A\mathbf{v} = \lambda\mathbf{v}$.
(1): $(A\mathbf{v})^*(A\mathbf{v}) = \mathbf{v}^*A^*A\mathbf{v} = \mathbf{v}^*I\mathbf{v} = |\mathbf{v}|^2$, on the other hand, $(A\mathbf{v})^*(A\mathbf{v}) = (\lambda\mathbf{v})^*(\lambda\mathbf{v}) = \overline{\lambda}\lambda\mathbf{v}^*\mathbf{v} = |\lambda|^2|\mathbf{v}|^2$, so $|\lambda|^2|\mathbf{v}|^2 = |\mathbf{v}|^2$, and since $\mathbf{v} \neq 0$, this implies $|\lambda|^2 = 1$, so $|\lambda| = 1$.
(2): $\mathbf{v}^*(A\mathbf{v}) = (\mathbf{v}^*A)\mathbf{v} = (\mathbf{v}^*A^*)\mathbf{v} = (A\mathbf{v})^*\mathbf{v} = (\lambda\mathbf{v})^*\mathbf{v} = \overline{\lambda}\mathbf{v}^*\mathbf{v} = \overline{\lambda}|\mathbf{v}|^2$, on the other hand, $\mathbf{v}^*(A\mathbf{v}) = \mathbf{v}^*(\lambda\mathbf{v}) = \lambda|\mathbf{v}|^2$, so $\overline{\lambda}|\mathbf{v}|^2 = \lambda|\mathbf{v}|^2$, and since $\mathbf{v} \neq 0$, this implies that $\overline{\lambda} = \lambda$, that is, $\lambda \in \mathbb{R}$.

Theorem:
If $A \in \mathbb{C}^{n\times n}$ is unitary, self-adjoint or normal, and $U \in \mathbb{C}^{n\times n}$ is unitary, then $U^{-1}AU$ is also unitary, self-adjoint or normal, respectively.

Proof: Let's notice first that $(U^{-1}AU)^* = (U^*AU)^* = U^*A^*U$.
If $A^* = A$, then $(U^{-1}AU)^* = U^*A^*U = U^{-1}AU$.
If $A^* = A^{-1}$, then $(U^{-1}AU)^* = U^*A^*U = U^{-1}A^{-1}U = (U^{-1}AU)^{-1}$.
Finally, $(U^{-1}AU)(U^{-1}AU)^* = (U^{-1}AU)(U^{-1}A^*U) = U^{-1}AA^*U$, and similary, $(U^{-1}AU)^*(U^{-1}AU) = U^{-1}A^*AU$, so if $AA^* = A^*A$, then $(U^{-1}AU)(U^{-1}AU)^* = (U^{-1}AU)^*(U^{-1}AU)$.

Spectral theorem

Theorem The following are equivalent for a matrix $A \in \mathbb{C}^{n\times n}$.
(i) A is normal.
(ii) There is a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^{-1}AU = U^*AU$ is diagonal.
(iii) There is an orthonormal basis in $\mathbb{C}^n$ consisting of eigenvectors of A.

The following two theorems are special cases of the spectral theorem.

Theorem: The following are equivalent for a matrix $A \in \mathbb{C}^{n\times n}$.
(i) A is self-adjoint.
(ii) There is a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^{-1}AU = U^*AU$ is real diagonal.
(iii) Every eigenvalue of A is real and there is an orthonormal basis in $\mathbb{C}^n$ consisting of eigenvectors of A.

And its version for real matrices:

Theorem: The following are equivalent for a matrix $A \in \mathbb{R}^{n\times n}$.
(i) A is symmetric.
(ii) There is an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that $U^{-1}AU = U^*AU$ is (real) diagonal.
(iii) There is an orthonormal basis in $\mathbb{R}^n$ consisting of eigenvectors of A.

Examples:
1. If A is the standard matrix of an orthogonal projection to a hyperplane, then it has an orthonormal basis of eigenvectors (an orthonormal basis of the hyperplane together with the normal vector of length 1), and the eigenvalues are 0 and 1, so A must be symmetric.
2. If A is the standard matrix of a projection to a plane in $\mathbb{R}^3$ along a vector which is not perpendicular to the given plane, then the eigenvector for 0 is not perpendicular to the eigenspace for 1, so the matrix cannot be symmetric.
3. If f is the rotation of $\mathbb{R}^2$ about the origin by the angle α, then its standard matrix is orthogonal: the orthonormal basis $\{\,\mathbf{i},\mathbf{j}\,\}$ is mapped to an orthonormal basis. (The matrix is $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.)