By using the Jordan normal form,

one can determine whether two matrices are similar;

for $J = P^{-1}AP$, the power J^m can be calculated relatively easily, so we also get A^m;

as in the case of the diagonal form (but a diagonal form does not always exist, even in $\mathbb{C}^{n\times n}$!) it provides a better understanding of the transformation.

(Note that in 1) and 3) we don't have to determine the transition matrix P.)

An immediate consequence of the Jordan normal form is the following condition for diagonalizability.

Theorem: A matrix $A \in K^{n\times n}$ whose characteristic polynomial can be written as a product of linear polynomials over K is diagonalizable if and only if the minimal polynomial has no multiple roots.

Proof: A diagonal matrix is also a Jordan matrix, and the Jordan matrix is essentially unique, so the matrix is diagonalizable if and only if every Jordan block is a 1×1 matrix, i.e. the maximal size of the λ_i-blocks is 1 for each i.

Exercise. Which of the following matrices are similar?

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ -2 & -3 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution. We know that the trace and the determinant are invariant under similarity. The traces of the above matrices are 0, 3, 3 and 3, so A is not similar to any of the others. The determinants of B, C and D are 0, 1, 1. so at most C and D can be similar. Let us determine the characteristic polinomials. $k_C(x) = k_D(x) = -(x-1)^3$, so both C and D have only 1 as their eigenvalue. It remains to determine the Jordan forms.

$$C - I = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -1 & 1 \\ -2 & -3 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$r(C-I)=2 \Rightarrow \dim V_1 = 3-2=1$ for C, so the Jordan form $\mathcal{J}(C)$ consists of one Jordan-1-block (and then it must be a block of size 3).

$$D-I=\begin{bmatrix}0&0&0\\2&0&0\\0&1&0\end{bmatrix} \mapsto \begin{bmatrix}2&0&0\\0&1&0\\0&0&0\end{bmatrix} \Rightarrow r(D-I)=2 \Rightarrow$$

$\dim V_1 = 3-2=1$ for D, so $\mathcal{J}(D)$ also consists of one Jordan-1-block. This means that C and D are similar to the same Jordan-matrix, consequently, they are similar to each other.

Exercise. What are the minimal polynomials of the matrices B, C and D of the previous exercise?
Solution Since the Jordan-form of C and D consists of one 1-block of size 3, the only root of their minimal polynomials is 1, and the exponent of $(x-1)$ is the size of the maximal 1-block, which is three. Thus $m_C(x)=m_D(x)=(x-1)^3$.
The rank of B can be immediately seen to be 1, so 0 is an eigenvalue, and $\dim V_0 = 3-1=2$, hence 0 has a multiplicity at least 2 in the characteristic polynomial. From the trace it follows that the third eigenvalue is 3. Thus the Jordan-form of B consists of three blocks: two for eigenvalue 0 and one for 3, so these blocks must of size 1, and then $m_B(x)=x(x-3)$ (B is diagonalizable).

Powers of a Jordan-matrix

Let $J\in\mathbb{C}^{m\times m}$ be a Jordan-block with eigenvalue λ. Then $J=\lambda I+N$, where N is a Jordan-block for eigenvalue 0 (a **nilpotent** Jordan-block). Since I (and also λI) commutes with every matrix, we can apply the binomial theorem to J^k:

$$J^k=(\lambda I+N)^k=\sum_{t=0}^{k}\binom{k}{t}\lambda^{k-t}N^t$$

$$=\begin{bmatrix}\lambda^k & \binom{k}{1}\lambda^{k-1} & \dots & \binom{k}{m-1}\lambda^{k-m+1}\\ 0 & \lambda^k & \dots & \binom{k}{m-2}\lambda^{k-m+2}\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & \lambda^k\end{bmatrix}$$

For a Jordan matrix $J=\operatorname{diag}(J_1,\dots,J_s)$ (meaning that the diagonal blocks are $J_1,\dots,J_s$ in this order), the power can be applied to each diagonal block: $J^k=\operatorname{diag}(J_1^k,\dots,J_s^k)$.

If for a matrix A, we also know a Jordan basis, that is, we have a transition matrix P such that $P^{-1}AP = J$ is a Jordan-matrix, then we can use this method for calculating the powers of A: $A^k = PJ^kP^{-1}$.

Limit of the powers of a matrix

Def. Let $A_k, A \in \mathbb{C}^{m\times n}$. Then $\lim\limits_{k\to\infty} A_k = A$ if $\lim\limits_{k\to\infty} (A_k)_{ij} = a_{ij}$ for all i, j, where $(A_k)_{ij}$ denotes the element of A_k at the (i, j) position.

Def. The **spectral radius** of a matrix $A \in \mathbb{C}^{n\times n}$ is

$$\rho(A) = \max\{\,|\lambda| \mid \lambda \text{ is an eigenvalue of } A\,\}.$$

Theorem. For a matrix $A \in \mathbb{C}^{n\times n}$

$$\lim_{k\to\infty} A^k = 0 \quad\Leftrightarrow\quad \rho(A) < 1$$

Proof. Let $J = P^{-1}AP$ be the Jordan form of A. Then $\rho(J) = \rho(A)$ and $\lim J^k = \lim P^{-1}A^kP = P^{-1}(\lim A^k)P = 0 \Leftrightarrow \lim A^k = 0$. So it suffices to prove the statement for Jordan matrices.
$\Rightarrow$: The powers of every Jordan λ-block converge to 0, so the diagonal elements $\lambda^k \to 0 \Rightarrow |\lambda| < 1$.
$\Leftarrow$: If J_0 is a Jordan λ-block for some $|\lambda| < 1$, then

$$|(J_0^k)_{t,t+i}| = \left|\binom{k}{i}\lambda^{k-i}\right| \le \frac{k^i|\lambda|^{k-i}}{i!} \to 0 \text{ if } k \to \infty,$$

so every element of J_0^k converges to 0.

Theorem. Let $A \in \mathbb{C}^{n\times n}$.
The sequence A^k is convergent if and only if

$\rho(A) < 1$, or
$\rho(A) = 1$ and $\lambda = 1$ is the only eigenvalue of absolute value 1, and all 1-blocks in the Jordan form are of size 1.

In the second case, for any vector $\mathbf{x} \in \mathbb{C}^n$, the limit $\mathbf{x}'$ of $A^k\mathbf{x}$ is either $\mathbf{0}$, or it is an eigenvector for $\lambda = 1$.

Proof. $\Rightarrow$: It is enough to prove the statement for Jordan matrices.
If there is a λ-block with $|\lambda| > 1$ or $\lambda \ne 1$ with $|\lambda| = 1$, then even the diagonal elements do not converge.
If there is a 1-block of size larger than 1, then the k'th power of its $(1, 2)$

element is k, and this diverges to ∞.

$\Leftarrow$: The conditions imply that the Jordan form of A is $P^{-1}AP = \begin{bmatrix} I_m & 0 \\ 0 & B \end{bmatrix}$, where B is a Jordan matrix with $\rho(B) < 1$. So $B^k \to 0$, and the first m columns of P span the eigenspace V_1. Thus

$$A^k\mathbf{x} = P \begin{bmatrix} I_m & 0 \\ 0 & B \end{bmatrix}^k P^{-1}\mathbf{x} \to P \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P^{-1}\mathbf{x} = [W \mid O]P^{-1}\mathbf{x}$$

is in the column space of W, i.e. in V_1.

(In fact, the limit $\mathbf{x}'$ is the projection of $\mathbf{x}$ onto V_1 along $\mathrm{Im}(A - I)$.)

Applying this to $\mathbf{x} = \mathbf{e}_1, \ldots, \mathbf{e}_n$, we also get the convergence of A^k.