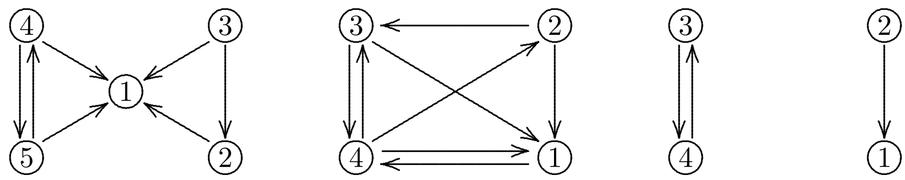
PageRank

1. Consider the following three graphs:



G_1 G_2 G_3

a) Determine the link matrices A_1, A_2 and A_3 of the three graphs defined in the PageRank algorithm. (If there is a vertex with no outgoing arrow, modify the graph so that you add an arrow from this vertex to all vertices of the graph, including itself.)
b) Find the solutions of the equation $\mathbf{x} = A_i\mathbf{x}$ for each i, to give a ranking of the pages/vertices of the graph.
c) Use the modified matrix $\hat{A}_3 = (1-p)A_3 + p\frac{1}{4}J$ to get a full ranking with $p = \frac{1}{4}$ and $p = \frac{1}{2}$, respectively.

Linear maps

2. Which of the following sets form a vector space over $\mathbb{R}$? Give a basis of the vector spaces.
a) 3×3 real upper triangular matrices with the usual operations;
b) invertible 2×2 real matrices;
c) polynomials of degree at most 4 which have -1 as one of their roots;

3. Choose a basis in the subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1 = (1,2,0,1)$, $\mathbf{v}_2 = (0,-1,1,-1)$, $\mathbf{v}_3 = (1,0,2,-1)$, $\mathbf{v}_4 = (0,1,1,1)$, $\mathbf{v}_5 = (2,3,3,1)$, and give the coordinate vectors of each $\mathbf{v}_i$ with respect to this basis.

4. Determine the matrices of the following linear maps with respect to the given basis or pair of bases:
a) rotation of the 3 dimensional space about the z axis by $90°$, in the standard basis;
b) $p(x) \mapsto (xp(x))'$ in the space of real polynomials of degree at most 2, in the standard basis $\{1, x, x^2\}$;
c) $\mathbf{x} \mapsto A\mathbf{x}$, where $A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$, $\mathcal{B} = \{(1,2),(1,1)\}$;
d) $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, where $\varphi(1,2,1) = (0,2,1)$, $\varphi(1,1,1) = (1,0,0)$, $\varphi(1,0,0) = (-1,0,0)$, in the standard basis;
e) $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(x,y) = (x+y, y, x)$, in the pair of bases $\mathcal{B}_1 = \{(1,1),(2,0)\}$, $\mathcal{B}_2 = \{(1,2,1),(-1,1,0),(0,1,1)\}$;
f) orthogonal projection onto the plane $x - 2y + z = 0$, in the standard basis;
g) transposition of 2×2 real matrices, in the standard basis.

5. Let A be the standard matrix of $f : (x,y,z) \longmapsto (x+y-2z, x+z, 2x+y-z, -x-z)$. Give bases for the null space of A (i.e. the kernel of f) and for the column space of A (i.e. the image of f).

6. Find a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that
a) $0 \neq \operatorname{Ker} f \subseteq \operatorname{Im} f$;
b) $\operatorname{Im} f$ is 2 dimesional, and f maps each vector of $\operatorname{Im} f$ into itself.

7. Prove that
a) $\operatorname{rank}(AB) \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$, where $A \in K^{k\times m}$ és $B \in K^{m\times n}$;
b) $|\operatorname{rank} A - \operatorname{rank} B| \leq \operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B$, where $A, B \in K^{m\times n}$.
(*Hint: Prove that, considering the matrices as linear maps in the natural way,* $\operatorname{Im} AB \leq \operatorname{Im} A$, $\operatorname{Ker} AB \geq \operatorname{Ker} B$ *and* $\operatorname{Im}(A+B) \leq \operatorname{span}(\operatorname{Im} A, \operatorname{Im} B)$.)

8. Show that for any matrix $A \in K^{m\times n}$ and any invertible matrices $B \in K^{m\times m}$ and $C \in K^{n\times n}$, we have $\operatorname{rank} BA = \operatorname{rank} AC = \operatorname{rank} A$.

9. Use Newton's interpolation to find a polynomial $f(x)$ of degree at most 3 such that $f(-1) = 0$, $f(0) = 1$, $f(2) = 1$ and $f(3) = -1$.

Eigenvectors, eigenvalues, diagonalization

10. Find the eigenvalues and eigenspaces of the following matrices. What is the action of the transformation $\mathbf{x} \mapsto D\mathbf{x}$ in $\mathbb{R}^3$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -4 & 1 \\ 1 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 1 & -3 \\ 0 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

11. Find the n'th power of the matrix $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

12. What are the eigenvalues and eigenvectors of the following linear transformations?

a) Rotation of $\mathbb{R}^3$ by $90°$ about the z axis.

b) Projection of $\mathbb{R}^2$ on the line $y = x$ in the direction of the vector $(1, 0)$.

c) $f : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ mapping every matrix A to $A + A^T$.

13. Show that every 3×3 real matrix has an eigenvector.

14. Prove that every eigenvector of A is an eigenvector of A^2. Is the reverse statement true?

Euclidean spaces and their transformations

15. Write the vector $\mathbf{b}$ as the sum of a vector which is orthogonal to $\mathbf{a}$ and a vector which is parallel to $\mathbf{a}$ if

a) $\mathbf{a} = (1, -2, 0, 1)$, $\mathbf{b} = (3, 1, 1, 1)$;

b) $\mathbf{a} = (1 + i, 1 - i)$, $\mathbf{b} = (i, 3 - i)$.

16. Prove that the subset $\{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 = x_4 + x_5\}$ is a hyperplane in $\mathbb{R}^5$, and determine its normal vector. Calculate the reflection of $(1, 0, 0, 0, 0)$ to this hyperplane.

17. Give the standard matrix of the orthogonal projection and of the reflection on the hyperplane $x + y - z = 0$ in $\mathbb{R}^3$.

18. Find the standard matrix of a reflection which maps the vector $(1, 2, -2)$ to $(3, 0, 0)$. (Hint: It is the reflection on the bisector plane of the line segment connecting the endpoints of the two vectors.)

19. Which of the following matrices are self-adjoint, unitary or normal?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} i & i \\ i & -i \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \qquad F = \begin{bmatrix} -1 & 2+i \\ 2-i & -5 \end{bmatrix} \qquad G = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & i \\ 1+i & 0 \end{bmatrix}$$

Orthogonalization, QR decomposition

20. a) Orthogonalize the vectors $\mathbf{b}_1 = (0, 1, -1, 0)$, $\mathbf{b}_2 = (1, 1, 0, -1)$, $\mathbf{b}_3 = (1, 2, 1, 0)$ in $\mathbb{R}^4$.

b) Orthogonalize the vectors $\mathbf{b}_1 = (i, 1, 0)$ and $\mathbf{b}_2 = (1 + i, 0, i)$ in $\mathbb{C}^3$, and then calculate the orthogonal projection of $\mathbf{v} = (1, 0, 0)$ on the subspace $\operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2\}$

21. Find a best approximate solution to the inconsistent system below, using the normal equations.

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

22. Prove that the normal system of equations $A^T A\mathbf{x} = A^T\mathbf{b}$ is consistent for any system $A\mathbf{x} = \mathbf{b}$.

23. Find the best approximate solution to the inconsistent system of equations below by first determining the QR decomposition of the coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 0 \\ -1 & 4 & 3 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

24. Consider the vectors $\mathbf{v}_1 = (1, 0, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0, 2)$, $\mathbf{v}_3 = (0, 0, 1, 1)$ in $\mathbb{R}^4$. Give an orthogonal basis of $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and of $W^\perp$.

25. Use the reduced QR decomposition of the coefficient matrix A in the solution of problem 23 to construct the full QR decomposition.

26. Determine the matrix of the Householder reflection and the Givens rotation mapping the vector $(-3, 0, 4)$ to $(5, 0, 0)$.

27. Find the full QR decomposition of the matrix A by using Householder reflections, and use this to give a reduced QR decomposition.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 2 & -6 \end{bmatrix}$$

28. Determine the QR decomposition of the matrix A, using Givens rotations, and in the end, if necessary, an extra reflection.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 4 & -5 \end{bmatrix}$$

Pseudoinverse

29. a) Show that for any matrices $A \in K^{m\times n}$ and $B \in K^{n\times m}$ the nonzero eigenvalues of $AB \in K^{m\times m}$ and $BA \in K^{n\times n}$ are the same.

b) Calculate the rank and eigenvalues of AA^T and $A^T A$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

30. Determine the pseudoinverses of the following matrices.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

31. Use the pseudoinverse calculated in problem 30 to find the smallest, best approximate solution of the system $y - z = 1$, $2x + y + z = 1$, $x + y = 0$.

32. The proof of the theorem showed that if $B \in \mathbb{R}^{m\times r}$ and $C \in \mathbb{R}^{r\times n}$ are of rank r, then $(BC)^+ = C^+B^+$. Show that the statement is not true in general for products of real matrices.

$$\text{For } B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad (BC)^+ \neq C^+B^+$$

Singular Value Decomposition

33. Determine the reduced and full SVD for the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

34. Calculate the pseudoinverse of B and C of problem 33 by using the reduced SVD.

35. Find polar decompositions of the square matrices in problem 33.

36. Find the best 1-rank approximation of the higher rank matrices of problem 33, using the reduced SVD. Calculate the error of the approximation, that is, $||A - A^{(1)}||$.

Jordan normal form

37. Is there a 3×3 matrix over $\mathbb{Q}$ with minimal polynomial

a) $x^2 - 2$; b) $x^2 + x$?

38. Suppose that A is a matrix over $\mathbb{C}$ such that $A^m = I$ for some $m \geq 1$. Prove that A is diagonalizable.

39. Which of the following matrices are diagonalizable over $\mathbb{C}$? Determine the Jordan normal form of the matrices.

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

40. What is the maximal number of non-similar complex matrices satisfying the following conditions? Give the Jordan normal form, the dimension of the eigenspaces and the minimal polinomial of the matrix in each possible case.

a) $k(x) = -x^5(x+1)^2, \ m(x) = x^3(x+1)$;

b) $k(x) = (x-1)^4x$, and the eigenspace for the eigenvalue 1 is 2-dimensional.

41. Find two non-similar 7×7 matrices which have the same minimal and characteristic polynomials, and their eigenspaces also have the same dimension.

Applications of the Jordan normal form

41. Which of the following matrices are similar?

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

42. Determine the J^{10} for the Jordan matrix

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

43. Is $\lim\limits_{k\to\infty} A^k$ convergent for the following matrices?

$$a)\ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad b)\ A = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} \qquad c)\ A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \qquad d)\ A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Nonnegative matrices

44. Which of the following matrices are irreducible, primitive, or stochastic?

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 0 \\ 1 & 1 & 1/4 & 1 \\ 0 & 0 & 1/4 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 5 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \frac{1}{2}\cdot\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

45. There is a flea on the number line, at first positioned randomly on any of the points 1, 2, 3 or 4. The flea changes its position in every second, always jumping to one of these four points. If it is on point 1 or 4 then it jumps to distance 1 with probability $\frac{2}{3}$, and to distance 2 with probability $\frac{1}{3}$. If it is on 2 or 3 then it jumps to one of the neighbouring numbers, each with probability $\frac{1}{2}$. What is the limit of the distribution of the position of the flea?