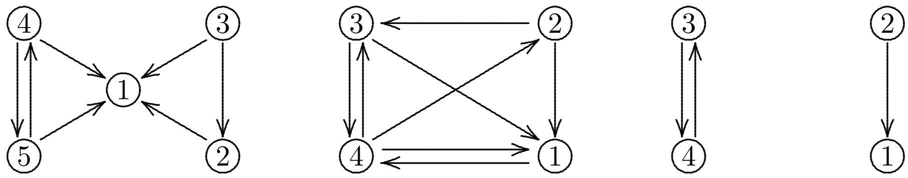
PageRank

1. *Consider the following three graphs:*



G_1 G_2 G_3

a) Determine the link matrices A_1, A_2 and A_3 of the three graphs defined in the PageRank algorithm. (If there is a vertex with no outgoing arrow, modify the graph so that you add an arrow from this vertex to all vertices of the graph, including itself.)

b) Find the solutions of the equation $\mathbf{x} = A_i\mathbf{x}$ for each i, to give a ranking of the pages/vertices of the graph.

c) Use the modified matrix $\hat{A}_3 = (1-p)A_3 + p\frac{1}{4}J$ to get a full ranking with $p = \frac{1}{4}$ and $p = \frac{1}{2}$, respectively.

Solution:

a) We need to modify G_1 and G_3 since their vertex 1 does not have outgoing arrows.

G_1' G_3'

Actually, we can fill the matrix column by column (without creating a modified graph) so that a_{ij} is the probability of going from v_i to v_j if we randomly choose an outgoing arrow, and if there is no such arrow then randomly choosing an arbitrary vertex.

$$A_1 = \begin{bmatrix} 1/5 & 1 & 1/2 & 1/2 & 1/2 \\ 1/5 & 0 & 1/2 & 0 & 0 \\ 1/5 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 0 & 1/2 \\ 1/5 & 0 & 0 & 1/2 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1/2 & 1/2 & 1/3 \\ 0 & 0 & 0 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 1 & 0 & 1/2 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1/4 & 1 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1 \\ 1/4 & 0 & 1 & 0 \end{bmatrix}$$

b) $\mathbf{x} = A\mathbf{x}$ is equivalent to the homogeneous system $(A - I)\mathbf{x} = \mathbf{0}$. Solve this by Gaussian elimination.

$$A_1 - I = \begin{bmatrix} -4/5 & 1 & 1/2 & 1/2 & 1/2 \\ 1/5 & -1 & 1/2 & 0 & 0 \\ 1/5 & 0 & -1 & 0 & 0 \\ 1/5 & 0 & 0 & -1 & 1/2 \\ 1/5 & 0 & 0 & 1/2 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1/5 & 0 & 0 & 1/2 & -1 \\ 1/5 & -1 & 1/2 & 0 & 0 \\ 1/5 & 0 & -1 & 0 & 0 \\ 1/5 & 0 & 0 & -1 & 1/2 \\ -4/5 & 1 & 1/2 & 1/2 & 1/2 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & 0 & 5/2 & -5 \\ 0 & -1 & 1/2 & -1/2 & 1 \\ 0 & 0 & -1 & -1/2 & 1 \\ 0 & 0 & 0 & -3/2 & 3/2 \\ 0 & 1 & 1/2 & 5/2 & -7/2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 5/2 & -5 \\ 0 & 1 & -1/2 & 1/2 & -1 \\ 0 & 0 & -1 & -1/2 & 1 \\ 0 & 0 & 0 & -3/2 & 3/2 \\ 0 & 0 & 1 & 2 & -5/2 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & 0 & 5/2 & -5 \\ 0 & 1 & 0 & 3/4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 \\ 0 & 0 & 0 & -3/2 & 3/2 \\ 0 & 0 & 0 & 3/2 & -3/2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & -5/2 \\ 0 & 1 & 0 & 0 & -3/4 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \cdot \begin{bmatrix} 5/2 \\ 3/4 \\ 1/2 \\ 1 \\ 1 \end{bmatrix},$$

so the for the ranks, $x_1 > x_2 > x_3 > x_4 = x_5$. (It can also be seen from the graph G_1 that the vertices v_4 and v_5 must have the same rank, since switching the labels 4 and 5 leaves the graph unchanged.)

$$A_2 - I = \begin{bmatrix} -1 & 1/2 & 1/2 & 1/3 \\ 0 & -1 & 0 & 1/3 \\ 0 & 1/2 & -1 & 1/3 \\ 1 & 0 & 1/2 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1/2 & -1/2 & -1/3 \\ 0 & -1 & 0 & 1/3 \\ 0 & 1/2 & -1 & 1/3 \\ 0 & 1/2 & 1 & -2/3 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & -1/2 & -1/2 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \cdot \begin{bmatrix} 2/3 \\ 1/3 \\ 1/2 \\ 1 \end{bmatrix},$$

so $x_4 > x_1 > x_3 > x_2$.

$$A_3 - I = \begin{bmatrix} -3/4 & 1 & 0 & 0 \\ 1/4 & -1 & 0 & 0 \\ 1/4 & 0 & -1 & 1 \\ 1/4 & 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1/4 & 0 & 1 & -1 \\ 1/4 & -1 & 0 & 0 \\ 1/4 & 0 & -1 & 1 \\ -3/4 & 1 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 4 & -4 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & 3 & -3 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

It is again clear from the symmetries of the graph that x_3 and x_4 must be equal but ranks shouldn't be 0 (actually one would expect $x_1 > x_2$). The problem is that in G_3' the surfer cannot get out of the subset $\{ v_3, v_4 \}$ once he got in. This is why we must use a modified $\hat{A}_3$ matrix as described in part c). The eigenspace for eigenvalue 1 of a positive stochastic matrix is always spanned by a positive eigenvector.

c) For $p = \frac{1}{4}$:

$$\hat{A}_3 = \begin{bmatrix} 1/4 & 13/16 & 1/16 & 1/16 \\ 1/4 & 1/16 & 1/16 & 1/16 \\ 1/4 & 1/16 & 1/16 & 13/16 \\ 1/4 & 1/16 & 13/16 & 1/16 \end{bmatrix}$$

$$\hat{A}_3 - I = \begin{bmatrix} -3/4 & 13/16 & 1/16 & 1/16 \\ 1/4 & -15/16 & 1/16 & 1/16 \\ 1/4 & 1/16 & 1/16 & 13/16 \\ 1/4 & 1/16 & 13/16 & 1/16 \end{bmatrix} \mapsto\mapsto \cdots \begin{bmatrix} 1 & 0 & 0 & -7/16 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \cdot \begin{bmatrix} 7/16 \\ 1/4 \\ 1 \\ 1 \end{bmatrix}.$$

For $p = \frac{1}{2}$:

$$\hat{A}_3 = \begin{bmatrix} 1/4 & 5/8 & 1/8 & 1/8 \\ 1/4 & 1/8 & 1/8 & 1/8 \\ 1/4 & 1/8 & 1/8 & 5/8 \\ 1/4 & 1/8 & 5/8 & 1/8 \end{bmatrix}$$

$$\hat{A}_3 - I = \begin{bmatrix} -3/4 & 5/8 & 1/8 & 1/8 \\ 1/4 & -7/8 & 1/8 & 1/8 \\ 1/4 & 1/8 & -7/8 & 5/8 \\ 1/4 & 1/8 & 5/8 & -7/8 \end{bmatrix} \mapsto\mapsto \cdots \begin{bmatrix} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \cdot \begin{bmatrix} 3/4 \\ 1/2 \\ 1 \\ 1 \end{bmatrix}$$

In both versions $x_4 = x_3 > x_1 > x_2$.

Linear maps

2. *Which of the following sets form a vector space over $\mathbb{R}$? Give a basis of the vector spaces.*

a) 3×3 *real upper triangular matrices with the usual operations;*
b) invertible 2×2 *real matrices;*
c) polynomials of degree at most 4 *which have* -1 *as one of their roots.*

Solution: a) It is a vector space because it is a subspace of $\mathbb{R}^{3\times 3}$: the sum or scalar multiple of upper triangular matrices is upper triangular. A basis is $\{\, E_{ij} \mid i \le j \,\} = \{\, E_{11}, E_{12}, E_{22}, E_{31}, E_{32}, E_{33} \,\}$, where E_{ij} is the matrix where the j'th element of the ith row is 1, and the others are 0.

b) Not a vector space. Though it is a subset of the vector space $\mathbb{R}^{2\times 2}$, it is not a subspace: 0 times an invertible matrix is the zero matrix, which is not invertible. But even if we add the 0 matrix to the subset we do not get a subspace: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

c) It is a vector space because it is a subspace of $\mathbb{R}[x]$ (the sum and scalar multiples of polynomials with root -1 also have -1 as a root and the degree does not increase). A basis is: $\{\, x+1,\ (x+1)x,\ (x+1)x^2,\ (x+1)x^3 \,\}$.

3. *Choose a basis in the subspace of* $\mathbb{R}^4$ *spanned by the vectors* $\mathbf{v}_1 = (1,2,0,1)$, $\mathbf{v}_2 = (0,-1,1,-1)$, $\mathbf{v}_3 = (1,0,2,-1)$, $\mathbf{v}_4 = (0,1,1,1)$, $\mathbf{v}_5 = (2,3,3,1)$, *and give the coordinate vectors of each* $\mathbf{v}_i$ *with respect to this basis.*

Solution: Bring the matrix A consisting of $\mathbf{v}_1,\dots,\mathbf{v}_5$ as columns to reduced row echelon form ($rref(A)$).

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 2 & -1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & -1 & -1 & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & -1 & -2 & 1 & -1 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & -1 & -2 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = rref(A)$$

Let us denote the columns of $rref(A)$ with $\mathbf{v}'_1,\dots,\mathbf{v}'_5$. Since the elementary row operations do not change the linear correspondances between the columns of a matrix, the maximal independent subset of the columns of $rref(A)$ tells the positions of colums of A which form a basis in the column space of A, i.e. the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1,\dots,\mathbf{v}_5$. The columns of the leading 1's of $rref(A)$ ($\mathbf{v}'_1 = \mathbf{e}_1$, $\mathbf{v}'_2 = \mathbf{e}_2$ and $\mathbf{v}'_4 = \mathbf{e}_3$) are clearly independent, generate all those vectors whose last entry is 0, and it can be seen from the colums of $rref(A)$ that $\mathbf{v}'_3 = \mathbf{e}_1 + 2\mathbf{e}_2 = \mathbf{v}'_1 + 2\mathbf{v}'_2$, while $\mathbf{v}'_5 = 2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3 = 2\mathbf{v}'_1 + 2\mathbf{v}'_2 + \mathbf{v}'_4$.

This means that $\mathcal{B} = \{\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4 \,\}$ (i.e. the set of columns of A at the positions of the columns of $rref(A)$ containing leading 1's) is a basis of $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\mathbf{v}_5\} \le \mathbb{R}^4$, and $\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2$, $\mathbf{v}_5 = 2\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4$ shows that the coordinate vectors are

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{v}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad [\mathbf{v}_4]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{v}_5]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

(In fact, they are simply the columns of $rref(A)$ if we cut off the zero rows.)

4. *Determine the matrices of the following linear maps with respect to the given basis or pair of bases:*

a) rotation of the 3 *dimensional space about the* z *axis by* 90°, *in the standard basis;*
b) $p(x) \mapsto (xp(x))'$ *in the space of real polynomials of degree at most* 2, *in the standard basis* $\{\, 1, x, x^2 \,\}$*;*

c) $\mathbf{x} \mapsto A\mathbf{x}$, *where* $A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$, $\mathcal{B} = \{(1,2),(1,1)\}$;

d) $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, *where* $\varphi(1,2,1) = (0,2,1)$, $\varphi(1,1,1) = (1,0,0)$, $\varphi(1,0,0) = (-1,0,0)$, *in the standard basis;*

e) $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(x,y) = (x+y, y, x)$, *in the pair of bases* $\mathcal{B}_1 = \{(1,1),(2,0)\}$, $\mathcal{B}_2 = \{(1,2,1),(-1,1,0),(0,1,1)\}$;

f) orthogonal projection onto the plane $x - 2y + z = 0$, *in the standard basis;*

g) transposition of 2×2 *real matrices, in the standard basis.*

Solution: a) $\mathbf{e}_1 \mapsto \mathbf{e}_2$, $\mathbf{e}_2 \mapsto -\mathbf{e}_1$, $\mathbf{e}_3 \mapsto \mathbf{e}_3$, so the standard matrix is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $a + bx + cx^2 \mapsto (ax + bx^2 + cx^3)' = a + 2bx + 3cx^2$, so the standard matrix is the matrix satisfying

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 2b \\ 3c \end{bmatrix} = \begin{bmatrix} 1a + 0b + 0c \\ 0a + 2b + 0c \\ 0a + 0b + 3c \end{bmatrix} \qquad \Rightarrow \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

c) Directly: $(1,2) \mapsto (-1,-2) = -1 \cdot (1,2) + 0 \cdot (1,1)$ gives the first column $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$,

and $(1,1) \mapsto (0,1) = 1 \cdot (1,2) - 1 \cdot (1,1)$ gives the second column $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so the matrix is

$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. Or using the transition matrix: $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

d) $A \cdot \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -2 \\ 0 & 2 \end{bmatrix}$

f) The projection of the vector (x,y,z) onto the normal vector $(1,-2,1)$ of the plane is $\dfrac{(x,y,z)(1,-2,1)}{|(1,-2,1)|^2}(1,-2,1) = \dfrac{1}{6} \cdot (x - 2y + z,\ -2x + 4y - 2z,\ x - 2y + z)$, so its projection onto the plane is

$(x,y,z) - \frac{1}{6}(x - 2y + z,\ -2x + 4y - 2z,\ x - 2y + z) = \frac{1}{6}(5x + 2y - z,\ 2x + 2y + 2z,\ -x + 2y + 5z)$,

and the standard matrix of the projection is $\frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$.

g) The action on the elements of the standard basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is $E_{11} \mapsto E_{11}$, $E_{12} \mapsto E_{21}$, $E_{21} \mapsto E_{12}$, $E_{22} \mapsto E_{22}$, so the standard matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

5. *Let* A *be the standard matrix of* $f : (x,y,z) \longmapsto (x + y - 2z, x + z, 2x + y - z, -x - z)$. *Give bases for the null space of* A *(i.e. the kernel of* f*) and for the column space of* A *(i.e. the image of* f*).*

Solution: The matrix and its reduced row echelon form are

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \mapsto \mapsto \mapsto \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the null space, i.e. the solution space of the homogeneous system of equations consists of the vectors $t\cdot\begin{bmatrix}-1\\3\\1\end{bmatrix}$ $(t\in\mathbb{R})$, so its basis is $\{\,[-1\quad 3\quad 1]^T\,\}$. The basis of the column space consists of the columns of A which stand in the same positions as the columns of $\mathrm{rref}(A)$ containing a leading 1, i.e. $\{\,[1\quad 1\quad 2\quad -1]^T,\ [1\quad 0\quad 1\quad 0]^T\,\}$.

6. *Find a linear transformation $f:\mathbb{R}^3\to\mathbb{R}^3$ such that*
a) $0\neq \operatorname{Ker} f\subseteq \operatorname{Im} f$*;*
b) $\operatorname{Im} f$ *is 2 dimesional, and f maps each vector of* $\operatorname{Im} f$ *into itself.*

Solution: It suffices the give the action of the transformations on the elements of a basis, say, on $\mathcal{B}=\{\,\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\,\}$ of $\mathbb{R}^3$.

a) By the dimension theorem, $\dim\operatorname{Ker} f+\dim\operatorname{Im} f=3$, and $1\le\dim\operatorname{Ker} f\le\dim\operatorname{Im} f$, so $\dim\operatorname{Ker} f=1$, and $\dim\operatorname{Im} f=2$. This can be achieved by a linear map acting on the basis elements as follows: $\mathbf{e}_1\mapsto\mathbf{0}$, $\mathbf{e}_2\mapsto\mathbf{e}_1$, $\mathbf{e}_3\mapsto\mathbf{e}_2$

b) We can aim to make $\operatorname{Im} f$ the subspace $\operatorname{span}(\mathbf{e}_1,\mathbf{e}_2)$. Since f is supposed to map every vector of $\operatorname{Im} f$ to itself, $f:\mathbf{e}_1\mapsto e_1$ and $f:\mathbf{e}_2\mapsto\mathbf{e}_2$. To make the image only two dimensional we only need to make the kernel nontrivial, say, let $f:\mathbf{e}_3\mapsto\mathbf{e}_3$. This defines f, and this f satisfies all the given conditions.

So the (standard) matrices of the maps constructed in part a) and b) are

$$\text{a)}\ \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\qquad\qquad \text{b)}\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

7. *Prove that*
a) $\operatorname{rank}(AB)\le\min\{\operatorname{rank}A,\operatorname{rank}B\}$, *where* $A\in K^{k\times m}$ *és* $B\in K^{m\times n}$*;*
b) $|\operatorname{rank}A-\operatorname{rank}B|\le\operatorname{rank}(A+B)\le\operatorname{rank}A+\operatorname{rank}B$, *where* $A,B\in K^{m\times n}$.
(Hint: Prove that, considering the matrices as linear maps in the natural way, $\operatorname{Im}AB\le\operatorname{Im}A$, $\operatorname{Ker}AB\ge\operatorname{Ker}B$ *and* $\operatorname{Im}(A+B)\le\operatorname{span}(\operatorname{Im}A,\ \operatorname{Im}B)$*.)*

Solution: For simplicity, we also denote by A and B the natural linear map corresponding to the matrices A and B, respectively: $A:\mathbf{x}\mapsto A\mathbf{x}$ and $B:\mathbf{x}\mapsto B\mathbf{x}$.

a) $(AB)\mathbf{x}=A(B\mathbf{x})\in\operatorname{Im}A$, so $\operatorname{Im}AB\le\operatorname{Im}A$, thus $\operatorname{rank}AB\le\operatorname{rank}A$. On the other hand, if $B\mathbf{x}=\mathbf{0}$, then $(AB)\mathbf{x}=A\mathbf{0}=\mathbf{0}$, so $\operatorname{Ker}B\le\operatorname{Ker}AB$, which implies by the dimension theorem that $\operatorname{rank}AB\le\operatorname{rank}B$. The two inequalities together prove the statement.

b) $\{(A+B)\mathbf{x}\,|\,\mathbf{x}\in K^n\}=\{\,A\mathbf{x}+B\mathbf{x}\,|\,\mathbf{x}\in K^n\,\}\le\operatorname{Im}A+\operatorname{Im}B$. In general, $\dim(U+W)\le\dim U+\dim W$ for the subspaces $U,W\le V$, since the union of the bases of U and W clearly spans $U+W:=\operatorname{span}(U,W)$, and its maximal independent subset will be a basis of $U+W$. This gives in our case that $\operatorname{rank}(A+B)\le\operatorname{rank}A+\operatorname{rank}B$. If we apply this for $A+B$ and $-B$, we get $\operatorname{rank}A=\operatorname{rank}((A+B)+(-B))\le\operatorname{rank}(A+B)+\operatorname{rank}(-B)=\operatorname{rank}(A+B)+\operatorname{rank}B$, implying that $\operatorname{rank}(A+B)\ge\operatorname{rank}A-\operatorname{rank}B$, and similarly, $\operatorname{rank}(A+B)\ge\operatorname{rank}B-\operatorname{rank}A$. The two together gives that $\operatorname{rank}(A+B)\ge|\operatorname{rank}A-\operatorname{rank}B|$.

8. *Show that for any matrix $A\in K^{m\times n}$ and any invertible matrices $B\in K^{m\times m}$ and $C\in K^{n\times n}$, we have* $\operatorname{rank}BA=\operatorname{rank}AC=\operatorname{rank}A$.

Solution: Problem 7.a) implies that $\operatorname{rank}BA\le\operatorname{rank}A$, and $\operatorname{rank}A=\operatorname{rank}B^{-1}(BA)\le\operatorname{rank}BA$, so $\operatorname{rank}BA=\operatorname{rank}A$, and similarly, $\operatorname{rank}AC=\operatorname{rank}A$.

9. *Use Newton's interpolation to find a polynomial $f(x)$ of degree at most 3 such that $f(-1)=0$, $f(0)=1$, $f(2)=1$ and $f(3)=-1$.*

Solution: The constant $f_0(x)\equiv0$ polynomial satisfies the first condition, $f_0(-1)=0$.
$f_1(x)=f_0(x)+a(x+1)$ still satisfies the first condition (since $a(x+1)=0$ at $x=-1$), and we can choose $a\in\mathbb{R}$ so that it also satisfies the second:

$$1=f_1(0)=0+a\cdot1\ \Leftrightarrow\ a=1\ \Leftrightarrow\ f_1(x)=0+(x+1)=x+1.$$

$f_2(x) = f_1(x) + b(x+1)x$ satisfies the first two conditions since $f_1(x)$ satisties them and the second term is 0 at -1 and 0. With the right choice of $b \in \mathbb{R}$, it also satisfies the third condition:

$$1 = f_2(2) = (2+1) + b \cdot 3 \cdot 2 \;\Leftrightarrow\; b = -\frac{1}{3} \;\Leftrightarrow\; f_2(x) = -\frac{1}{3}x^2 + \frac{2}{3}x + 1.$$

Finally, $f_3(x) = f_2(x) + c(x+1)x(x-2)$ satisfies the first three conditions since $f_2(x)$ did, and the second term is 0 at $-1, 0, 2$, and it also satisfies the fourth if we choose c appropriately:

$$-1 = f_3(3) = -\frac{1}{3} \cdot 3^2 + \frac{2}{3} \cdot 3 + 1 + c \cdot 4 \cdot 3 \cdot 1 \;\Leftrightarrow\; c = -\frac{1}{12} \;\Leftrightarrow\; f_3(x) = -\frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{5}{6}x + 1.$$

So the unique f of degree at most 3 satisfying all four conditions is $f(x) = -\frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{5}{6}x + 1$.

Eigenvectors, eigenvalues, diagonalization

10. *Find the eigenvalues and eigenspaces of the following matrices. What is the action of the transformation* $\mathbf{x} \mapsto D\mathbf{x}$ *in* $\mathbb{R}^3$*.*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -4 & 1 \\ 1 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 1 & -3 \\ 0 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

Solution: $k_A(x) = x^2 - 1 \Rightarrow$ The eigenvalues of A are ± 1.
Eigenspace for

$$\lambda = 1: \quad \left[\begin{array}{rr|r} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \mapsto \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenspace for

$$\lambda = -1: \quad \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So the basis of the eigenspace V_1 is $\{(1,1)\}$ and of V_{-1} is $\{(-1,1)\}$.
$k_B(x) = x^2 + 1$ has no real roots, so B as a real matrix has no eigenvalues. However if we consider B as a complex matrix, then its eigenvalues are $\pm i$.
The eigenspace for

$$\lambda = i: \quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \mapsto \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \;\Rightarrow\; \mathbf{x} = \begin{bmatrix} it \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix},$$

and similarly, the elements of the eigenspace for $\lambda = -i$ are $t \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

$$k_C(x) = \begin{vmatrix} 4-x & -4 & 1 \\ 1 & -1-x & 0 \\ -2 & 4 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & -1-x \\ -2 & 4 \end{vmatrix} + (1-x) \begin{vmatrix} 4-x & -4 \\ 1 & -1-x \end{vmatrix} =$$

$$= (2-2x) + (1-x)(x^2-3x) = -(x-1)(x^2-3x+2) = -(x-1)^2(x-2),$$

so the eigenvalues are $1,\ 1,\ 2$.
Eigenspace for

$$\lambda = 1: \quad \begin{bmatrix} 3 & -4 & 1 \\ 1 & -2 & 0 \\ -2 & 4 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 0 \\ 3 & -4 & 1 \\ -2 & 4 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Eigenspace for

$$\lambda = 2: \quad \begin{bmatrix} 2 & -4 & 1 \\ 1 & -3 & 0 \\ -2 & 4 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3 & 0 \\ 2 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Since there are no three independent eigenvectors, C is not diagonalizable.

$$k_D(x) = \begin{vmatrix} 3-x & 1 & -3 \\ 0 & 1-x & 0 \\ 2 & 1 & -2-x \end{vmatrix} = (1-x) \begin{vmatrix} 3-x & -3 \\ 2 & -2-x \end{vmatrix} = -(x-1)(x^2-x) = -(x-1)^2 x,$$

so the eigenvalues are 1, 1, 0.
Eigenspace for

$$\lambda = 1: \quad \begin{bmatrix} 2 & 1 & -3 \\ 0 & 0 & 0 \\ 2 & 1 & -3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{rcl} x_1 & = & -\frac{1}{2}s + \frac{3}{2}t \\ x_2 & = & s \\ x_3 & = & t \end{array}, \quad \textit{i.e.} \quad \mathbf{x} = \frac{s}{2} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \frac{t}{2} \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix},$$

so the basis of the eigenspace V_1 is $\{(-1,2,0),(3,0,2)\}$.
Eigenspace for

$$\lambda = 0: \quad \begin{bmatrix} 3 & 1 & -3 \\ 0 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This means, that $\mathcal{B} = \{(-1,2,0),(3,0,2),(1,0,1)\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors of D, so the matrix D is diagonalizable. The diagonal form also shows that the transformation $\mathbf{x} \mapsto D\mathbf{x}$ maps every vector of the plane $\operatorname{span}(\mathbf{b}_1, \mathbf{b}_2)$ to itself, while it maps $\mathbf{b}_3$ to $\mathbf{0}$, so it is a projection on the plane in the direction of $\mathbf{b}_3$.

11. *Find the n'th power of the matrix* $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

Solution: $k_A(x) = \begin{vmatrix} 1-x & -2 \\ -2 & 1-x \end{vmatrix} = x^2 - 2x - 3 = (x-3)(x+1)$, so the eigenvalues are 3 and -1.
Eigenspace for

$$\lambda = 3: \quad \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenspace for

$$\lambda = -1: \quad \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So

$$A = PDP^{-1}, \quad \text{where} \quad P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$A^n = PD^nP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3^n + (-1)^n & -3^n + (-1)^n \\ -3^n + (-1)^n & 3^n + (-1)^n \end{bmatrix}$$

12. *What are the eigenvalues and eigenvectors of the following linear transformations?*
a) Rotation of $\mathbb{R}^3$ *by* $90°$ *about the z axis.*
b) Projection of $\mathbb{R}^2$ *on the line* $y = x$ *in the direction of the vector* $(1,0)$.
c) $f : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ *mapping every matrix* A *to* $A + A^T$.

Solution: a) The only eigenvectors are the ones parallel with the z axis, i.e. $t(0,0,1)$ for $t \neq 0$, with eigenvalue 1.

b) The eigenvectors are the nonzero multiples of $(1,1)$ with eigenvalue 1, and the nonzero multiples of $(1,0)$ with eigenvalue 0.

c) Since $A+A^T$ is a symmetric matrix ($(A+A^T)^T = A^T + A = A + A^T$), a matrix can only be an eigenvector if either it is symmetric, or its image is 0, that is, $A^T = -A$, which means that it is antisymmetric. In the first case $f(A) = A + A^T = 2A$, so the eigenvalue is 2, and the eigenspace is $V_2 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a,b,c \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ a 3-dimensional subspace of $\mathbb{R}^{2\times 2}$. In the second case the eigenvalue is 0, and $V_0 = \left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \mid a \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is 1-dimensional.

Just for practice, calculate the eigenvalues and eigenspaces from the standard matrix of the transformation. The standard basis is $\mathcal{E} = \{ E_{11}, E_{12}, E_{21}, E_{22} \}$, and

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix},$$

so for $M = [f]_{\mathcal{E}}$

$$M \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2a \\ b+c \\ b+c \\ 2d \end{bmatrix} \quad \Rightarrow \quad M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Then $|M - xI| = (2-x)^2(x^2 - 2x) = (x-2)^3 x \Rightarrow$ the eigenvalues are 2, 2, 2, 0 and the eigenvectors for

$$\lambda = 2: \quad \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x} = \begin{bmatrix} s \\ t \\ t \\ u \end{bmatrix},$$

so the basis of the eigenspace V_2 is $\{ E_{11},\ E_{12} + E_{21},\ E_{22} \}$, while for

$$\lambda = 0: \quad \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix},$$

so the basis of the eigenspace V_0 is $\{ -E_{12} + E_{21} \}$.

13. *Show that every 3×3 real matrix has an eigenvector.*

Solution: The characteristic polynomial $-x^3 + a_2x^2 + a_1x + a_0$ must have at least one real root (since its limit in $-\infty$ is $+\infty$, and in $+\infty$ is $-\infty$). So the matrix has a real eigenvalue λ, and for the real eigenvalue we can find real eigenvectors by solving the system of equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$ over $\mathbb{R}$.

14. *Prove that every eigenvector of A is an eigenvector of A^2. Is the reverse statement true?*

Solution: If $\mathbf{v}$ is an eigenvector for the eigenvalue λ, then $A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$. The reverse statement is usually not true, for example the rotation of the plane about the origin by 90° has no real eigenvector, while for its square every nonzero vector is an eigenvector.

Euclidean spaces and their transformations

15. *Write the vector* $\mathbf{b}$ *as the sum of a vector which is orthogonal to* $\mathbf{a}$ *and a vector which is parallel to* $\mathbf{a}$ *if*

a) $\mathbf{a} = (1, -2, 0, 1)$, $\mathbf{b} = (3, 1, 1, 1)$;

b) $\mathbf{a} = (1 + i, 1 - i)$, $\mathbf{b} = (i, 3 - i)$.

Solution: a) The component parallel with $\mathbf{a}$ is the orthogonal projection of $\mathbf{b}$ on $\mathbf{a}$, which is $\mathbf{b}' = \dfrac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}|^2}\mathbf{a} = \dfrac{2}{6}\mathbf{a} = \left(\dfrac{1}{3}, -\dfrac{2}{3}, 0, \dfrac{1}{3}\right)$, and the component orthogonal to $\mathbf{a}$ is $\mathbf{b} - \mathbf{b}' = \left(\dfrac{8}{3}, \dfrac{5}{3}, 1, \dfrac{2}{3}\right)$.

b) For the orthogonal projection $\mathbf{b}' = \dfrac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}|^2}\mathbf{a}$, we need $\langle \mathbf{a}, \mathbf{b}\rangle = (1 - i)i + (1 + i)(3 - i) = 5 + 3i$ and $|\mathbf{a}|^2 = |1 + i|^2 + |1 - i|^2 = 1^2 + 1^2 + 1^2 + 1^2 = 4$. So the component parallel with $\mathbf{a}$ is $\mathbf{b}' = \dfrac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}|^2}\mathbf{a} = \dfrac{5 + 3i}{4}\mathbf{a} = \left(\dfrac{1}{2} + 2i,\ 2 - \dfrac{1}{2}i\right)$, and the component orthogonal to $\mathbf{a}$ is $\mathbf{b} - \mathbf{b}' = \left(-\dfrac{1}{2} - i,\ 1 - \dfrac{1}{2}i\right)$.

16. *Prove that the subset* $\{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 = x_4 + x_5\}$ *is a hyperplane in* $\mathbb{R}^5$*, and determine its normal vector. Calculate the reflection of* $(1, 0, 0, 0, 0)$ *to this hyperplane.*

Solution: The subset consists of those vectors $\mathbf{x}$ whose scalar product with $(1, 1, 0, -1, -1)$ is 0, so it is the hyperplane with normal vector $\mathbf{a} = (1, 1, 0, -1, -1)$. We can obtain the reflection of a vector $\mathbf{v}$ on the hyperplane as $\mathbf{v} - \frac{2}{|\mathbf{a}|^2}\langle \mathbf{a}, \mathbf{v}\rangle\mathbf{a}$, so the reflection of $\mathbf{v} = (1, 0, 0, 0, 0)$ is $\mathbf{w} = (1, 0, 0, 0, 0) - \frac{2}{4}(1, 1, 0, -1, -1) = (\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$.
Indeed, the connecting vector is $\mathbf{v} - \mathbf{w} = (\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2})$ is parallel to $\mathbf{a}$, so it is perpendicular to the hyperplane, and the midpoint of the connecting segment is $\frac{1}{2}(\mathbf{v} + \mathbf{w}) = (\frac{3}{4}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4})$ is in the hyperplane.

17. *Give the standard matrix of the orthogonal projection and of the reflection on the hyperplane* $x + y - z = 0$ *in* $\mathbb{R}^3$.

Solution: The normal vector of the plane is $(1, 1, -1)$, so the standard matrix of the orthogonal projection is $I - \frac{1}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*$, where $\mathbf{a}$ is the normal vector written as a column vector:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The standard matrix of the reflection is $I - \frac{2}{|\mathbf{a}|^2}\mathbf{a}\mathbf{a}^*$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

18. *Find the standard matrix of a reflection which maps the vector* $(1, 2, -2)$ *to* $(3, 0, 0)$. *(Hint: It is the reflection on the bisector plane of the line segment connecting the endpoints of the two vectors.)*

Solution: The bisector plane has a normal vector $(3, 0, 0) - (1, 2, -2) = (2, -2, 2)$, or its scalar multiple, $(1, -1, 1)$, and the plane contains the origin, since $|(1, 2, -2)| = 3 = |(3, 0, 0)|$. So it is a hyperplane $H(\mathbf{a})$ with $\mathbf{a} = [1 \quad -1 \quad 1]^T$. Thus the standard matrix of the reflection is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}.$$

One can easily check that this transformation maps $(1, 2, -2)$ to $(3, 0, 0)$.

19. *Which of the following matrices are self-adjoint, unitary or normal?*

$$A=\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}\qquad B=\begin{bmatrix}0&2&-1\\-2&0&3\\1&-3&0\end{bmatrix}\qquad C=\begin{bmatrix}i&i\\i&-i\end{bmatrix}\qquad D=\begin{bmatrix}2&1\\3&2\end{bmatrix}$$

$$E=\begin{bmatrix}1&-1\\-1&3\end{bmatrix}\qquad F=\begin{bmatrix}-1&2+i\\2-i&-5\end{bmatrix}\qquad G=\begin{bmatrix}1/3&-2/3&-2/3\\2/3&2/3&-1/3\\2/3&-1/3&2/3\end{bmatrix}\qquad H=\begin{bmatrix}1&i\\1+i&0\end{bmatrix}$$

Solution: A, E, F are self-adjoint, G is unitary. $B^*=-B$, so $B^*B=BB^*$, that is, B is normal. $C=i\begin{bmatrix}1&1\\1&-1\end{bmatrix}$ is a scalar multiple of a self-adjoint (so also normal) matrix, thus C is normal. (If M is normal, then $(cM)^*(cM)=|c|^2M^*M=|c|^2MM^*=(cM)(cM)^*$.) D and H are not even normal, because $D^*D\neq DD^*$ and $H^*H\neq HH^*$.

Orthogonalization, QR decomposition

20. *a) Orthogonalize the vectors* $\mathbf{b}_1=(0,1,-1,0)$, $\mathbf{b}_2=(1,1,0,-1)$, $\mathbf{b}_3=(1,2,1,0)$ *in* $\mathbb{R}^4$.
b) Orthogonalize the vectors $\mathbf{b}_1=(i,1,0)$ *and* $\mathbf{b}_2=(1+i,0,i)$ *in* $\mathbb{C}^3$*, and then calculate the orthogonal projection of* $\mathbf{v}=(1,0,0)$ *on the subspace* $\operatorname{span}\{\,\mathbf{b}_1,\mathbf{b}_2\,\}$

Solution: a) $\mathbf{c}_1=(0,1,-1,0)$.
$\mathbf{c}_2=\mathbf{b}_2-\frac{\langle\,\mathbf{c}_1,\mathbf{b}_2\,\rangle}{|\mathbf{c}_1|^2}\mathbf{c}_1=(1,1,0,-1)-\frac{1}{2}(0,1,-1,0)=(1,\frac{1}{2},\frac{1}{2},-1)$.
Use the parallel $\tilde{\mathbf{c}}_2=(2,1,1,-2)$ instead.
$\mathbf{c}_3=\mathbf{b}_3-\frac{\langle\,\mathbf{c}_1,\mathbf{b}_3\,\rangle}{|\mathbf{c}_1|^2}\mathbf{c}_1-\frac{\langle\,\tilde{\mathbf{c}}_2,\mathbf{b}_3\,\rangle}{|\tilde{\mathbf{c}}_2|^2}\tilde{\mathbf{c}}_2=(1,2,1,0)-\frac{1}{2}(0,1,-1,0)-\frac{5}{10}(2,1,1,-2)=(0,1,1,1)$.
So the corresponding orthogonal vectors are: $(0,1,-1,0)$, $(2,1,1,-2)$, $(0,1,1,1)$ and the orthonormal vectors are $\frac{1}{\sqrt{2}}(0,1,-1,0)$, $\frac{1}{\sqrt{10}}(2,1,1,-2)$, $\frac{1}{\sqrt{3}}(0,1,1,1)$.

b) $\mathbf{c}_1=(i,1,0)$,
$\mathbf{c}_2=(1+i,0,i)-\frac{-i(1+i)+1\cdot 0+0\cdot i}{1^2+1^2+0^2}(i,1,0)=(1+i,0,i)-\frac{1-i}{2}(i,1,0)=(\frac{1}{2}+\frac{1}{2}i,-\frac{1}{2}+\frac{1}{2}i,i)$, or we can take $\tilde{\mathbf{c}}_2=(1+i,-1+i,2i)$ instead.
The projection of $\mathbf{v}$ on $\operatorname{span}\{\,\mathbf{b}_1,\mathbf{b}_2\,\}=\operatorname{span}\{\,\mathbf{c}_1,\tilde{\mathbf{c}}_2\,\}$ is
$\frac{\langle\,\mathbf{v},\mathbf{c}_1\,\rangle}{|\mathbf{c}_1|^2}\mathbf{c}_1-\frac{\langle\,\tilde{\mathbf{c}}_2,\mathbf{v}\,\rangle}{|\tilde{\mathbf{c}}_2|^2}\tilde{\mathbf{c}}_2=\frac{-i}{2}(i,1,0)+\frac{1-i}{8}(1+i,-1+i,2i)=$
$(\frac{1}{2},-\frac{1}{2}i,0)+(\frac{1}{4},\frac{1}{4}i,\frac{1}{4}+\frac{1}{4}i)=(\frac{3}{4},-\frac{1}{4}i,\frac{1}{4}+\frac{1}{4}i)$.

21. *Find a best approximate solution to the inconsistent system below, using the normal equations.*

$$\begin{bmatrix}1&2&-1\\-1&1&1\\-1&0&1\end{bmatrix}\mathbf{x}=\begin{bmatrix}1\\0\\2\end{bmatrix}$$

Solution:

$$\begin{bmatrix}1&-1&-1\\2&1&0\\-1&1&1\end{bmatrix}\left[\begin{array}{ccc|c}1&2&-1&1\\-1&1&1&0\\-1&0&1&2\end{array}\right]=\left[\begin{array}{ccc|c}3&1&-3&-1\\1&5&-1&2\\-3&-1&3&1\end{array}\right]\mapsto$$

$$\left[\begin{array}{ccc|c}1&5&-1&2\\3&1&-3&-1\\0&0&0&0\end{array}\right]\mapsto\left[\begin{array}{ccc|c}1&5&-1&2\\0&-14&0&-7\\0&0&0&0\end{array}\right]\left[\begin{array}{ccc|c}1&0&-1&-\frac{1}{2}\\0&1&0&\frac{1}{2}\\0&0&0&0\end{array}\right]\Rightarrow\mathbf{x}=\begin{bmatrix}-\frac{1}{2}+t\\\frac{1}{2}\\t\end{bmatrix}$$

This gives a best approximate solution for any $t\in\mathbb{R}$, for instance, $t=0$ gives $\mathbf{x}=(-\frac{1}{2},\frac{1}{2},0)$.

22. *Prove that the normal system of equations* $A^TA\mathbf{x}=A^T\mathbf{b}$ *is consistent for any system* $A\mathbf{x}=\mathbf{b}$.

Solution: Let $A \in \mathbb{R}^{m\times n}$. We are going to prove that $\operatorname{Im} A^T = \operatorname{Im} A^T A$, so any vector in $\operatorname{Im} A^T$, that is, any vector of the form $A^T\mathbf{b}$ can be written as $A^T A\mathbf{x}$ for some $\mathbf{x}$.
It is clear that $\operatorname{Im} A^T A \le \operatorname{Im} A^T$ (since $A^T A\mathbf{x} = A^T(A\mathbf{x})$), so we only need to show that $\dim \operatorname{Im} A^T A = \dim \operatorname{Im} A^T$, or equivalently, $\operatorname{rank} A^T A = \operatorname{rank} A^T$, and the latter is known to be equal to $\operatorname{rank} A$.

So it is enough to prove that $\operatorname{rank} A^T A = \operatorname{rank} A$. By the dimension theorem, $\operatorname{rank} A^T A = n - \dim \operatorname{Ker} A^T A$, and $\operatorname{rank} A = n - \dim \operatorname{Ker} A$, so we only need to show that $\dim \operatorname{Ker} A = \dim \operatorname{Ker} A^T A$. In fact, it is true that $\operatorname{Ker} A = \operatorname{Ker} A^T A$. Indeed, $\operatorname{Ker} A \le \operatorname{Ker} A^T A$, since $A\mathbf{x} = \mathbf{0}$ implies $A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$, on the other hand, if $A^T A\mathbf{x} = \mathbf{0}$, then $0 = \mathbf{x}^T A^T A\mathbf{x} = |A\mathbf{x}|^2 \Rightarrow A\mathbf{x} = \mathbf{0}$. This finishes the proof.

23. *Find the best approximate solution to the inconsistent system of equations below by first determining the QR decomposition of the coefficient matrix.*

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 0 \\ -1 & 4 & 3 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Solution: Let us determine the QR decomposition by using the Gram–Schmidt method for the column vectors $\mathbf{v}_1 = (1,0,-1,-1,-1)$, $\mathbf{v}_2 = (0,-2,4,0,0)$, $\mathbf{v}_3 = (5,0,3,-1,-1)$.
$\mathbf{c}_1 = (1,0,-1,-1,-1)$.
$\mathbf{c}_2 = (0,-2,4,0,0) - \frac{-4}{4}(1,0,-1,-1,-1) = (1,-2,3,-1,-1)$
$\mathbf{c}_3 = (5,0,3,-1,-1) - \frac{4}{4}(1,0,-1,-1,-1) - \frac{16}{16}(1,-2,3,-1,-1) = (3,2,1,1,1)$.
Then the corresponding orthonormal vectors are:
$\mathbf{q}_1 = \frac{1}{2}(1,0,-1,-1,-1)$, $\mathbf{q}_2 = \frac{1}{4}(1,-2,3,-1,-1)$, $\mathbf{q}_3 = \frac{1}{4}(3,2,1,1,1)$.
(When orthogonalizing for QR decomposition, make sure that, in case you modify the $\mathbf{c}_i$ vectors in the process, you only multiply them with positive scalars so that in the end you get an R with positive diagonal elements.)
Then $Q = [\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3]$, and $R = Q^T A$, where A is the coefficient matrix.

$$Q = \begin{bmatrix} 1/2 & 1/4 & 3/4 \\ 0 & -2/4 & 2/4 \\ -1/2 & 3/4 & 1/4 \\ -1/2 & -1/4 & 1/4 \\ -1/2 & -1/4 & 1/4 \end{bmatrix}, \quad \text{so } A = QR \text{ with}$$

$$R = Q^T A = \frac{1}{4}\begin{bmatrix} 2 & 0 & -2 & -2 & -2 \\ 1 & -2 & 3 & -1 & -1 \\ 3 & 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 0 \\ -1 & 4 & 3 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

The best approximate solution of the given system of equations is the solution of the equation $R\mathbf{x} = Q^T\mathbf{b}$:

$$\left[\begin{array}{ccc|c} 2 & -2 & 2 & -1/2 \\ 0 & 4 & 4 & 3/4 \\ 0 & 0 & 4 & 5/4 \end{array}\right] \Rightarrow z = 5/16,\ y = -1/8,\ x = -11/16$$

24. *Consider the vectors* $\mathbf{v}_1 = (1,0,-1,1)$, $\mathbf{v}_2 = (1,0,0,2)$, $\mathbf{v}_3 = (0,0,1,1)$ *in* $\mathbb{R}^4$. *Give an orthogonal basis of* $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *and of* $W^\perp$.

Solution: Complete the set to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$, that is,

$$(1,0,-1,1),(1,0,0,2),(0,0,1,1),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1).$$

We get $\mathbf{c}_1 = (1,0,-1,1)$,
$\mathbf{c}_2 = (1,0,0,2) - \frac{3}{3}(1,0,-1,1) = (0,0,1,1)$,
$\mathbf{c}_3 = (0,0,1,1) - 0\mathbf{c}_1 - \frac{2}{2}(0,0,1,1) = \mathbf{0}$, so we discard this $\mathbf{c}_3$,
and we get an orthogonal basis $\{(1,0,-1,1),(0,0,1,1)\}$ of W. Continuing with the $\mathbf{e}_i$ vectors we get the new
$\mathbf{c}_3 = (1,0,0,0) - \frac{1}{3}(1,0,-1,1) - 0(0,0,1,1) = (\frac{2}{3},0,\frac{1}{3},-\frac{1}{3})$, or rather the parallel
$\tilde{\mathbf{c}}_3 = (2,0,1,-1)$.
$\mathbf{c}_4 = (0,1,0,0) - 0(1,0,-1,1) - 0(0,0,1,1) - 0(2,0,1,-1) = (0,1,0,0)$.
This is the fourth nonzero orthogonal vector in $\mathbb{R}^4$, so this completes the basis, and thus $\{(2,0,1,-1),(0,1,0,0)\}$ is an orthogonal basis of $W^\perp$.
(Actually, we could have discarded $\mathbf{v}_3$ after seeing that it is the same as $\mathbf{c}_2$, and could have also saved work by noticing that $(0,1,0,0)$ is orthogonal to all the previous vectors, but this solution shows how the algorithm handles even the redundant vectors or the orthogonal ones.)

25. *Use the reduced QR decomposition of the coefficient matrix A in problem 23 to construct the full QR decomposition.*

Solution: We have to complete the columns of Q, or more conveniently the corresponding orthogonal system $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ to an orthogonal basis of $\mathbb{R}^5$. Add as many as needed of $\mathbf{e}_1, \dots, \mathbf{e}_5$ until we get two more extra nonzero orthogonal vectors, and then we normalize the new vectors as well.
$\mathbf{c}_1 = (1,0,-1,-1,-1)$
$\mathbf{c}_2 = (1,-2,3,-1,-1)$
$\mathbf{c}_3 = (3,2,1,1,1)$
$\mathbf{c}_4 = (1,0,0,0) - \frac{1}{4}(1,0,-1,-1,-1) - \frac{1}{16}(1,-2,3,-1,-1) - \frac{3}{16}(3,2,1,1,1) = (\frac{1}{8},-\frac{1}{4},-\frac{1}{8},\frac{1}{8},\frac{1}{8})$, or rather the parallel
$\tilde{\mathbf{c}}_4 = (1,-2,-1,1,1)$.
We may notice that $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ are all in the hyperplane $x_4 = x_5$. So to get a full basis of $\mathbb{R}^5$, we have to add a vector outside this hyperplane, for instance, $\mathbf{e}_4$.
$\mathbf{c}_5 = (0,0,0,1,0) + \frac{1}{4}(1,0,-1,-1,-1) + \frac{1}{16}(1,-2,3,-1,-1) - \frac{1}{16}(3,2,1,1,1) - \frac{1}{8}(1,-2,-1,1,1) = (0,0,0,\frac{1}{2},-\frac{1}{2})$, or rather the parallel
$\tilde{\mathbf{c}}_5 = (0,0,0,1,-1)$.
(Actually this last one can also be obtained as the normal vector of the hyperplane $x_4 - x_5 = 0$, because that is clearly orthogonal to the previous four vectors.)
By dividing the orthogonal vectors by their lengths, we get the columns of the orthogonal matrix $\hat{Q}$, and we complete R with the necessary number of zero rows.

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 0 \\ -1 & 4 & 3 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 3/4 & 1/2\sqrt{2} & 0 \\ 0 & -2/4 & 2/4 & -2/2\sqrt{2} & 0 \\ -1/2 & 3/4 & 1/4 & -1/2\sqrt{2} & 0 \\ -1/2 & -1/4 & 1/4 & 1/2\sqrt{2} & 1/\sqrt{2} \\ -1/2 & -1/4 & 1/4 & 1/2\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

26. *Determine the matrix of the Householder reflection and the Givens rotation mapping the vector $(-3,0,4)$ to $(5,0,0)$.*

Solution: The Householder-reflection will be across the hyperplane with normal vector $(8,0,-4)$, or simpler, $\mathbf{a} = (2,0,-1)$, and its matrix is

$$I - \frac{2}{|\mathbf{a}|^2}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix}.$$

The Givens rotation rotates the xz plane, and leaves the y axis fixed. Its matrix is

$$\begin{bmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{bmatrix}.$$

27. *Find the full QR decomposition of the matrix A by using Householder reflections, and use this to give a reduced QR decomposition.*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 2 & -6 \end{bmatrix}$$

Solution: The first reflection must map $(2,1,2)$ to $(3,0,0)$, so it is a reflection across the hyperplane with normal vector $(1,-1,-2)$.

$$Q_1A = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 0 & 3 \\ 0 & 4 \end{bmatrix}$$

For the next transformation we must find first the reflection in $\mathbb{R}^2$, which maps the lower part of the second column, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Its matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$, and Q_2 will be the matrix which acts on the second and third coordinate as this matrix, while leaves the first coordinate unchanged.

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \end{bmatrix}, \text{ so } \quad Q_2Q_1A = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 0 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \hat{R}$$

So $A = \hat{Q}\hat{R}$, where $\hat{Q} = Q_1^TQ_2^T (= Q_1Q_2)$, and we get the reduced QR decomposition by cutting off the last column(s) of $\hat{Q}$ to the shape of A and the zero row(s) of $\hat{R}$.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 2 & -6 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 10 & 11 & -2 \\ 5 & -2 & 14 \\ 10 & -10 & -5 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 10 & 11 \\ 5 & -2 \\ 10 & -10 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 0 & 5 \end{bmatrix}$$

28. *Determine the QR decomposition of the matrix A, using Givens rotations, and in the end, if necessary, an extra reflection.*

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 4 & -5 \end{bmatrix}$$

Solution: The first rotation will fix the first two elements of the first column: it takes $(0,3)$ to $(3,0)$, while the third coordinat remains unchanged.

$$Q_1A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 4 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 1 & -1 \\ 4 & 4 & -5 \end{bmatrix}$$

Now we fix the first and third element of the first column: $(3,4)$ should go to $(5,0)$, while the middle component remains unchanged.

$$Q_2Q_1A = \frac{1}{5}\begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 \\ 0 & 1 & -1 \\ 4 & 4 & -5 \end{bmatrix} = \begin{bmatrix} 5 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

Luckily, Q_2Q_1A is already an upper triangular matrix, so we do not have to work on the second column. However, the last element of the diagonal is not positive so we add a reflection accross the xy-plane.

$$Q_3Q_2Q_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 5 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} = \hat{R}.$$

So $A = \hat{Q}\hat{R}$, where $\hat{Q} = Q_1^TQ_2^TQ_3^T$, which gives the decomposition

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 4 & -5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 0 & -5 & 0 \\ 3 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} 5 & 5 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Pseudoinverse

29. *a) Show that for any matrices $A \in K^{m\times n}$ and $B \in K^{n\times m}$ the nonzero eigenvalues of $AB \in K^{m\times m}$ and $BA \in K^{n\times n}$ are the same.*
b) Calculate the rank and eigenvalues of AA^T and A^TA for the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: a) Suppose $\mathbf{0} \neq \mathbf{v} \in K^m$ is an eigenvector of AB for an eigenvalue $\lambda \neq 0$, that is, $AB\mathbf{v} = \lambda\mathbf{v}$. Then $AB\mathbf{v} \neq \mathbf{0}$, and $BA(B\mathbf{v}) = B(AB\mathbf{v}) = B\lambda\mathbf{v} = \lambda B\mathbf{v}$, and $B\mathbf{v} \neq \mathbf{0}$, since $AB\mathbf{v} \neq \mathbf{0}$. This means that $B\mathbf{v}$ is an eigenvector of BA with eigenvalue λ, so λ is also an eigenvalue of BA. Switching the roles of A and B, we also get that all the nonzero eigenvalues of BA are eigenvalues of AB.

b) We have seen that $r(A^TA) = r(A)$, so $r(AA^T) = r(A^T) = r(A)$, so both ranks are $r(A) = 2$. According to part a), the nonzero eigenvalues are the same for A^TA and AA^T, and the ranks show that 0 is an eigenvalue of the 3×3 matrix, AA^T but not an eigenvalue of A^TA. Let's check this by caluclating both.

$k_{A^TA}(x) = \begin{vmatrix} 6-x & 1 \\ 1 & 2-x \end{vmatrix} = x^2 - 8x + 11$, so the eigenvalues are $4 \pm \sqrt{5}$.

$k_{AA^T}(x) = \begin{vmatrix} 1-x & 2 & -1 \\ 2 & 5-x & -1 \\ -1 & -1 & 2-x \end{vmatrix} = -x^3 + 8x^2 - 11x = -x(x^2 - 8x + 11)$, so the eigenvalues are 0 and $4 \pm \sqrt{5}$.

30. *Determine the pseudoinverses of the following matrices.*

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution: If a matrix B is of full column rank then $B^+ = (B^TB)^{-1}B^T$, if a matrix C is of full row rank then $C^+ = C^T(CC^T)^{-1}$. If A is not a full rank matrix then the rank factorization $A = BC$ yields a matrix B of full column rank and C of full row rank such that $A^+ = C^+B^+$.

a) $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is of full row rank, so $C^+ = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} =$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}.$$

b) $C = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ is of full column rank, so $C^+ = \left(\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$.

c) $A = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} \mapsto \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \Rightarrow A = BC = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} \Rightarrow$

$C^+ = \begin{bmatrix}1\\1\end{bmatrix}\left(\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)^{-1} = \dfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$, $B^+ = \left(\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 2\end{bmatrix} = \dfrac{1}{5}\begin{bmatrix}1 & 2\end{bmatrix} \Rightarrow$

$A^+ = C^+B^+ = \dfrac{1}{10}\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}$.

d) $\begin{bmatrix}0 & 1 & -1\\2 & 1 & 1\\1 & 1 & 0\end{bmatrix} \mapsto \begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\\2 & 1 & 1\end{bmatrix} \mapsto \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix} \Rightarrow A = BC = \begin{bmatrix}0 & 1\\2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\end{bmatrix} \Rightarrow$

$C^+ = \begin{bmatrix}1 & 0\\0 & 1\\1 & -1\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\1 & -1\end{bmatrix}\right)^{-1} = \begin{bmatrix}1 & 0\\0 & 1\\1 & -1\end{bmatrix}\frac{1}{3}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 1\\1 & 2\\1 & -1\end{bmatrix}$,

$B^+ = \left(\begin{bmatrix}0 & 2 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 1\\1 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 2 & 1\\1 & 1 & 1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}3 & -3\\-3 & 5\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}-3 & 3 & 0\\5 & -1 & 2\end{bmatrix}$

$\Rightarrow A^+ = C^+B^+ = \frac{1}{18}\begin{bmatrix}-1 & 5 & 2\\7 & 1 & 4\\-8 & 4 & -2\end{bmatrix}$.

31. *Use the pseudoinverse calculated in problem 30 to find the smallest, best approximate solution of the system $y - z = 1$, $2x + y + z = 1$, $x + y = 0$.*

Solution: The coefficient matrix is the one in problem 30, part d.

$$A^+\mathbf{b} = \frac{1}{18}\begin{bmatrix}-1 & 5 & 2\\7 & 1 & 4\\-8 & 4 & -2\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2/9\\4/9\\-2/9\end{bmatrix}.$$

(One can check that $A[\frac{2}{9}\ \frac{4}{9}\ -\frac{2}{9}]^T = \frac{2}{3}[1\ 1\ 1]^T$ is indeed the orthogonal projection of (1,1,0) on the column space of the coefficient matrix: it is in the column space, and $(1,1,0)-\frac{2}{3}(1,1,1) = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ is orthogonal to all three columns of A.)

32. *The proof of the theorem showed that if $B \in \mathbb{R}^{m\times r}$ and $C \in \mathbb{R}^{r\times n}$ are of rank r, then $(BC)^+ = C^+B^+$. Show that the statement is not true in general for products of real matrices.*

$$\text{For } B = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \text{ and } C = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix},\quad (BC)^+ \neq C^+B^+$$

Solution: The rank factorization $B = \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$ gives $B^+ = \begin{bmatrix}1 & 1\end{bmatrix}^+\begin{bmatrix}1\\1\end{bmatrix}^+ = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$, and $C^+ = C^{-1} = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$, so $C^+B^+ = \frac{1}{4}\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}$.

On the other hand, $BC = \begin{bmatrix}1 & 3\\1 & 3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 3\end{bmatrix} \Rightarrow (BC)^+ = \frac{1}{10}\begin{bmatrix}1\\3\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{20}\begin{bmatrix}1 & 1\\3 & 3\end{bmatrix}$.

Singular Value Decomposition

33. *Determine the reduced and full SVD for the following matrices*

$A = \begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$ $\qquad B = \begin{bmatrix}-1 & -1\\-1 & -1\end{bmatrix}$ $\qquad C = \begin{bmatrix}1 & 2 & 0\\0 & -2 & 1\end{bmatrix}$

Solution: a) $A^TA = A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow \lambda_1 = 4,\ \lambda_2 = 1,\ \sigma_1 = 2,\ \sigma_2 = 1,\ \Sigma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

The unit eigenvectors of A^TA are:

for $\lambda = 4$: $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for $\lambda = 1$: $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$AV_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \Rightarrow$

$U_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which we obtained by dividing the columns of AV_1 by $\sigma_1 = 2$ and $\sigma_2 = 1$, respectively (in the general case we would divide them by $\sigma_1, \dots, \sigma_r$, where $r = r(A)$).

So $A = U_1\Sigma_1V_1^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = U\Sigma V^T$ is both the reduced and the full SVD, since A is an invertible square matrix.

b) $B^TB = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $k_{B^TB}(x) = x^2 - 4x$, $\lambda_1 = 4$, $\lambda_2 = 0$, $\sigma_1 = 2$ is the only singular value, $r = 1$, $\Sigma_1 = [2]$. Eigenvectors for $\lambda_1 = 4$: $t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, unit eigenvector: $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow V_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$BV_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \end{bmatrix} \Rightarrow U_1 = \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

So $B = U_1\Sigma_1V_1^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix} [2] \frac{1}{\sqrt{2}} [1 \ \ 1]$ is the reduced SVD.

We obtain the full SVD by extending U_1 and V_1 to orthogonal (square) matrices by adding extra orthonormal columns, and extending Σ_1 to Σ of size equal to the size of the original matrix by adding extra zeros:

$U = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

$B = U\Sigma V^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is a full SVD.

c) $C^TC = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 8 & -2 \\ 0 & -2 & 1 \end{bmatrix}$, $k_{C^TC}(x) = -x^3 + 10x^2 - 9x = -x(x-9)(x-1) \Rightarrow$

$\lambda_1 = 9$, $\lambda_2 = 1$, $\lambda_3 = 0$, $\sigma_1 = 3$, $\sigma_2 = 1$, $r = 2$, $\Sigma_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 9$:

$$\begin{bmatrix} -8 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & -8 \end{bmatrix} \mapsto \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 4 \\ -8 & 2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 4 \\ -8 & 0 & -8 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow t\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$$

for $\lambda = 1$:

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 7 & -2 \\ 0 & -2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \text{ So}$$

$$\mathbf{v}_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix},\ \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ V_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 & 3 \\ -4 & 0 \\ 1 & 3 \end{bmatrix},\ CV_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -3 & 1 \\ 3 & 1 \end{bmatrix},\ U_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} = U_1\Sigma_1V_1^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3\sqrt{2}} \begin{bmatrix} -1 & -4 & 1 \\ 3 & 0 & 3 \end{bmatrix}$ is a reduced SVD of C.

To get a full SVD, of the semiorthogonal matrices, we only have to complete V, since U_1 is already orthogonal. We can do it, for instance, by adding the cross product of the two columns, $\frac{1}{3}[-2\ 1\ 2]^T$ as a third column.

$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} = U\Sigma V^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1/3\sqrt{2} & -4/3\sqrt{2} & 1/3\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -2/3 & 1/3 & 2/3 \end{bmatrix}$ is a full SVD of C.

Remark: The full SVD (and then the reduced SVD) of A and B can be obtained easier, if we use the fact that these are symmetric matrices, so we can diagonalize them by an orthogonal matrix, and then we only have to modify one of them to make the diagonal elements of the middle diagonal matrix nonnegative:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = QDQ^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T =$$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, *where we obtained the columns of Q as an orthonormal eigenbasis of A in the decreasing order of the absolute values of the eigenvalues. The modifying matrix* $\operatorname{diag}(\pm 1, \dots, \pm 1)$ *is orthogonal, so the new right side matrix will also be orthogonal.*

As for C, it is somewhat easier to calculate the reduced SVD of C^T, because in that case we only have to find the eigenvalues and eigenvectors of the 2×2 matrix CC^T, and then we get the SVD of C as the transposed of the SVD of C^T.

34. *Calculate the pseudoinverse of B and C of problem 33 by using the reduced SVD.*

Solution: $B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = U_1\Sigma_1 V_1^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix}[2]\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} \Rightarrow$

$B^+ = V_1\Sigma_1^{-1}U_1^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}[\tfrac{1}{2}]\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$.

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} = U_1\Sigma_1 V_1^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{3\sqrt{2}}\begin{bmatrix} -1 & -4 & 1 \\ 3 & 0 & 3 \end{bmatrix} \Rightarrow$$

$$C^+ = V_1\Sigma_1^{-1}U_1^T = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 & 3 \\ -4 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 10 & 8 \\ 4 & -4 \\ 8 & 10 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 & 4 \\ 2 & -2 \\ 4 & 5 \end{bmatrix}$$

35. *Find polar decompositions of the square matrices in problem 33.*

Solution: The polar decomposition of $A = U\Sigma V^T$ is $A = (U\Sigma U^T)(UV^T)$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

36. *Find the best 1-rank approximation of the higher rank matrices of problem 33, using the reduced SVD. Calculate the error of the approximation, that is, $||A - A^{(1)}||$.*

Solution: $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow$

$A^{(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}[\,2\,]\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$

$A - A^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \|A - A^{(1)}\| = \sqrt{1^2 + 0^2 + 0^2 + 0^2} = \sqrt{1} = 1.$

$$C=\begin{bmatrix}1 & 2 & 0\\ 0 & -2 & 1\end{bmatrix}=U_1\Sigma_1V_1^T=\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\frac{1}{3\sqrt{2}}\begin{bmatrix}-1 & -4 & 1\\ 3 & 0 & 3\end{bmatrix}\Rightarrow$$

$$C^{(1)}=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}[\,3\,]\frac{1}{3\sqrt{2}}[-1\quad -4\quad 1]=\begin{bmatrix}1/2 & 2 & -1/2\\ -1/2 & -2 & 1/2\end{bmatrix}$$

$$C-C^{(1)}=\begin{bmatrix}1/2 & 0 & 1/2\\ 1/2 & 0 & 1/2\end{bmatrix}\Rightarrow \|C-C^{(1)}\|=\sqrt{\tfrac14+\tfrac14+\tfrac14+\tfrac14}=1.$$

Jordan normal form

37. *Is there a 3×3 matrix over $\mathbb{Q}$ with minimal polynomial*
a) x^2-2; *b)* x^2+x?

Solution: a) If $m_A(x)=x^2-2=(x-\sqrt{2})(x+\sqrt{2})$, then the only eigenvalues of A are $\sqrt{2}$ and $-\sqrt{2}$. But then $k_A(x)=-(x-\sqrt{2})^{a_1}(x+\sqrt{2})^{a_2}$, where $a_1,a_2\geq 1$, and $a_1+a_2=3$. So $k_A(x)$ is either $-m(x)(x-\sqrt{2})=-(x^2-2)(x-\sqrt{2})=-x^3+\sqrt{2}x^2+2x-2\sqrt{2}$ or $k_A(x)=-m(x)(x+\sqrt{2})=-(x^2-2)(x+\sqrt{2})=-x^3-\sqrt{2}x^2-2x+2\sqrt{2}$, but neither of them is in $\mathbb{Q}[x]$. So there is no such matrix.

b) Since $m_A(x)=x^2+x=x(x+1)$, the eigenvalues are 0 and -1. Furthermore, both of them have multiplicity 1 in the minimal polinomial, so the matrix A is diagonalizable. This condition is satisfied by an arbitrary diagonal matrix which has only 0s and -1s in its diagonal. Actually every 3×3 matrix with minimal polynomial x^2+x is similar to one of the diagonal matrices with diagonal elements $0,0,-1$ or $0,-1,-1$.

38. *Suppose that A is a matrix over $\mathbb{C}$ such that $A^m=I$ for some $m\geq 1$. Prove that A is diagonalizable.*

Solution: Since A is a 'root' of the polynomial x^m-1, the minimal polynomial is a divisor of x^m-1. But x^m-1 has m different roots in $\mathbb{C}$ (the m complex mth roots of unity), so the minimal polynomial has no multiple roots. Thus the matrix A is diagonalizable.

39. *Which of the following matrices are diagonalizable over $\mathbb{C}$? Determine the Jordan normal form of the matrices.*

$$A=\begin{bmatrix}-3 & 1 & 2\\ 1 & 1 & 0\\ 2 & 0 & -1\end{bmatrix}\qquad B=\begin{bmatrix}0 & 0 & -2\\ 1 & 0 & 3\\ 0 & 1 & 0\end{bmatrix}\qquad C=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 1 & 0\\ 0 & 0 & 3 & 1\\ 0 & 0 & 0 & 4\end{bmatrix}\qquad D=\begin{bmatrix}0 & 0 & 0 & 0\\ -1 & 0 & 0 & 2\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Solution: The characteristic polynomial of the matrix A is $-x^3-3x^2+6x=-x(x+\frac32-\frac{\sqrt{33}}{2})(x+\frac32+\frac{\sqrt{33}}{2})$. A is diagonalizable because it has 3 different eigenvalues. The Jordan normal form in this case is any of the diagonal matrices with the eigenvalues 0, $-\frac32+\frac{\sqrt{33}}{2}$ and $-\frac32-\frac{\sqrt{33}}{2}$ in its diagonal, in some order.

$k_B(x)=-x^3+3x-2=-(x-1)^2(x+2)$, so the eigenvalues are 1 and -2. The eigenspace corresponding to the eigenvalue 1 is the solution space of the equation $(B-I)\mathbf{x}=\mathbf{0}$, but it is 1-dimensional, so the Jordan normal form has only one 1-block, whose size is 2, and one -2-block of size 1.

$$B\sim\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Since the Jordan normal form is not diagonal, the matrix is not diagonalizable.

C is a triangular matrix, so without any further calculation we can see that its eigenvalues are the diagonal elements $1,2,3$ and 4. Since C is a 4×4 matrix with four different eigenvalues, C must be diagonalizable, and its Jordan normal form is the diagonal matrix with $1,2,3,4$ in the diagonal.

$k_D(x)=x^4$, so 0 is the only eigenvalue. The Jordan normal form of D contains only Jordan blocks corresponding to 0. The rank of D is 2, thus the eigenspace corresponding to 0 is $4-2=2$-dimensional, consequently the Jordan normal form has only two Jordan blocks. $D^2=0$, so the largest Jordan block is of size 2. Hence the Jordan normal form is the block diagonal matrix with two diagonal blocks, each equal to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$:

$$D \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the Jordan form is not diagonal, D is not diagonalizable.

40. *What is the maximal number of non-similar complex matrices satisfying the following conditions? Give the Jordan normal form, the dimension of the eigenspaces and the minimal polinomial of the matrix in each possible case.*

a) $k(x) = -x^5(x+1)^2, \;\; m(x) = x^3(x+1)$;

b) $k(x) = (x-1)^4 x$, *and the eigenspace for the eigenvalue* 1 *is* 2*-dimensional.*

Solution: a) We know from the characteristic polynomial that the Jordan normal form consists of 0- and -1-blocks, the sum of the sizes of the former is 5, of the latter is 2. From the minimal polynomial we can deduce that the largest 0-block is of size 3, and the largest -1-block has size 1. So there are only two possibilities: the diagonal blocks of the Jordan normal form are:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, [0], \; [0], \; [-1], \; [-1] \qquad \text{or} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; [-1] \; [-1]$$

The minimal polynomial was given, the dimension of the eigenspace for -1 is 2 (since there are two Jordan (-1)-blocks), and for 0 it is 3 in the first case, and 2 in the second.

b) By the characteristic polynomial, the diagonal blocks of the Jordan normal form can only be 1-blocks and one 1×1 0-block. The sum of the sizes of the 1-blocks is 4, and we know by the dimension of the eigenspace that there are exactly 2 1-blocks. So there are two possibilities:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, [1], \; [0] \qquad \text{or} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; [0]$$

Here the minimal polynomials are $(x-1)^3 x$ and $(x-1)^2 x$, respectively.

41. *Find two non-similar* 7×7 *matrices which have the same minimal and characteristic polynomials, and their eigenspaces also have the same dimension.*

Solution: We can take the two Jordan-matrices, with diagonal blocks

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, [0], \qquad \text{and} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In both cases 0 is the only eigenvalue, the minimal polynomial is x^3, and the eigenspace is 3-dimensional.

Applications of the Jordan normal form

41. *Which of the following matrices are similar?*

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Solution: The traces of all four matrices are 4, but the determinants of $A, B.C$ are 2, while $\det D = 0$, so D is not similar to any of the others. The characteristic polynomials of A, B and C

are $-(x-1)^2(x-2)$, so their Jordan normal form is either diagonal, or it consists of a 2×2 1-block and a 1×1 2-block. It is enough to check the dimension of the eigenspace for the eigenvalue 1 to decide whether there are 2 or 1 1-blocks in the Jordan form.

$$A-I=\begin{bmatrix}0&0&-1\\0&0&2\\0&0&1\end{bmatrix}\qquad B-I=\begin{bmatrix}0&0&0\\2&1&0\\-1&0&0\end{bmatrix}\qquad C-I=\begin{bmatrix}1&0&0\\1&-1&1\\-1&-1&1\end{bmatrix}$$

It is easy to see that $r(A-I)=1$ and $r(B-I)=r(C-I)=2$, so the eigenspace for 1 is 2-dimensional for A, and 1-dimensional for B and C. So the Jordan forms

$$A\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix},\qquad B\sim\begin{bmatrix}1&1&0\\0&1&0\\0&0&2\end{bmatrix}\sim C,$$

show that B and C are similar, and A is not similar to them.

42. *Determine the J^{10} for the Jordan matrix*

$$J=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$$

Solution: $J^{10}=\begin{bmatrix}2^{10}&\binom{10}{1}2^9&\binom{10}{2}2^8\\0&2^{10}&\binom{10}{1}2^9\\0&0&2^{10}\end{bmatrix}=\begin{bmatrix}2^{10}&10\cdot 2^9&45\cdot 2^8\\0&2^{10}&10\cdot 2^9\\0&0&2^{10}\end{bmatrix}$

43. *Is $\lim\limits_{k\to\infty}A^k$ convergent for the following matrices?*

a) $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ *b)* $A=\begin{bmatrix}3/5&4/5\\-4/5&3/5\end{bmatrix}$ *c)* $A=\begin{bmatrix}0&1\\-1&2\end{bmatrix}$ *d)* $A=\begin{bmatrix}1&0&-1\\0&1&2\\0&0&\frac{1}{2}\end{bmatrix}$

Solution: a) $k_A(x)=x^2-5x-2\Rightarrow\lambda_{1,2}=\frac{5\pm\sqrt{33}}{2}$, and $\frac{5+\sqrt{33}}{2}>1$, so $\rho(A)>1\Rightarrow\lim A^k$ is not convergent.

b) This is an orthogonal matrix, so every eigenvalue has absolute value 1. But the characteristic polynomial $k_A(x)=x^2-\frac{6}{5}x+1$ shows that these eigenvalues are different from 1, so $\lim A^k$ is not convergent.

c) $k_A(x)=x^2-2x+1=(x-1)^2$, so 1 is the only eigenvalue. However, the eigenspace is only 1-dimensional, so the Jordan form of A is $\begin{bmatrix}1&1\\0&1\end{bmatrix}$, containing a 1-block of size larger than 1, so $\lim A^k$ is not convergent.

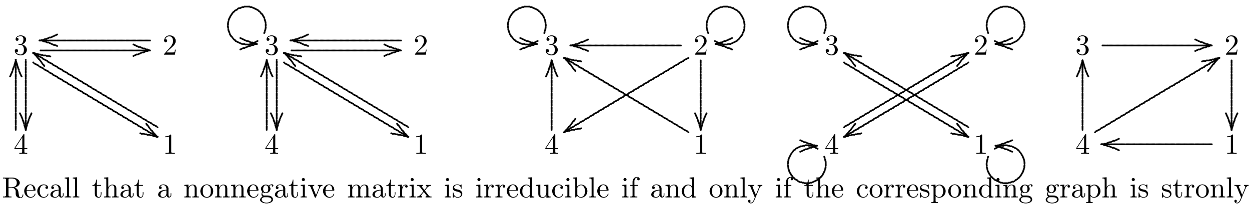
d) The eigenvalues of A are $1,1,\frac{1}{2}$, so $\rho(A)\le 1$. The only eigenvalue with absolute value 1 is 1 itself. Finally, $r(A-I)=1$, so the eigenspace for 1 is 2-dimensional, showing that there are two 1-blocks in the Jordan form, thus all the 1-blocks are of size 1. The three conditions together imply that $\lim A^k$ is convergent.

Nonnegative matrices

44. *Which of the following matrices are irreducible, primitive, or stochastic?*

$$\begin{bmatrix}0&0&1&0\\0&0&2&0\\1&3&0&2\\0&0&1&0\end{bmatrix}\quad\begin{bmatrix}0&0&1/4&0\\0&0&1/4&0\\1&1&1/4&1\\0&0&1/4&0\end{bmatrix}\quad\begin{bmatrix}0&0&5&0\\1&1&1&1\\0&0&2&0\\0&0&1&0\end{bmatrix}\quad\frac{1}{2}\cdot\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\quad\begin{bmatrix}0&0&0&3\\1&0&0&0\\0&2&0&0\\0&1&1&0\end{bmatrix}$$

Solution: All of the five matrices are nonnegative but only the the second and the fourth are stochastic because there the sum of the elements in each column is 1. To determine whether they are irreducible or stochastic, we need to draw the graphs associated to the matrices.



Recall that a nonnegative matrix is irreducible if and only if the corresponding graph is stronly connected. And the matrix is primitive if it is irreducible, and the greatest common divisor of the lengthes of the cycles in the graph is 1.

The first graph is strongly connected: one can go from each point to each point (even to itself) through 3. So the matrix is irreducible. But it is not primitive because every cycle is even.

The second graph has an extra loop compared to the first. So the matrix is still irreducible, and also primitive because it has a cycle of length 1.

The third is not irreducible since one cannot go from 3 to any other point. Consequently, it is not primitiv, either.

The fourth is disconnected, so the matrix is not irreducible and not primitive.

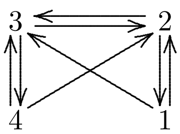
The fifth is irreducible since one can go from any point to any point along the cycle 4–3–2–1–4. Furthermore, it has cycles of length 4 and 3, so the greatest common divisor of the cycle lengthes is 1. Thus the fifth matrix is also primitive.

45. *There is a flea on the number line, at first positioned randomly on any of the points* 1, 2, 3 *or* 4. *The flea changes its position in every second, always jumping to one of these four points. If it is on point* 1 *or* 4 *then it jumps to distance* 1 *with probability* $\frac{2}{3}$, *and to distance* 2 *with probability* $\frac{1}{3}$. *If it is on* 2 *or* 3 *then it jumps to one of the neighbouring numbers, each with probability* $\frac{1}{2}$. *What is the limit of the distribution of the position of the flea?*

Solution: The transition matrix for the distribution of the position of the flea is

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

The graph associated to this matrix is:



This shows that the matrix is irreducible, since one can go to any point to any point along the cycle 1–3–4–2–1, and it is also primitive because it has cycles of length 3 and 2 (for example, 2–3–4–2 and 2–1–2).

It is easy to show that the theorem about the convergence of positive stochastic matrices also holds for primitive stochastic matrices. So $\lim\limits_{k\to\infty} A^k \frac{1}{4}\mathbf{1}$ exists, and it is the only stochastic vector in the 1-dimensional eigenspace for eigenvalue 1. What remains is calculating this eigenvector.

$$A - I = \begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 \\ \frac{2}{3} & -1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & -1 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & -1 \end{bmatrix} \mapsto \begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{2}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & -1 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & -1 \end{bmatrix} \mapsto \begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{2}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & \frac{4}{3} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = t \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

This eigenvector is a stochastic vector if $6t = 1$, so the limit is

$$\begin{bmatrix} 1/6 \\ 2/6 \\ 2/6 \\ 1/6 \end{bmatrix}.$$