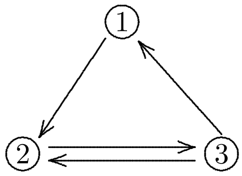
Applied algebra **Midterm Test** *2020, Fall semester*

Show all the details of your work in your solution. You may use a calculator or a calculator application, and you are free to look at the lecture notes or the exercises and their solutions. But do not communicate with each other or other people during the test. You can reach me if necessary through Teams chat or email: `lukacs@math.bme.hu`.

1. *Determine the link matrix A of the following graph defined in the PageRank algorithm. Find a ranking as a positive eigenvector of matrix A for eigenvalue* 1. *(4 marks)*



Solution: $A = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$ is the link matrix. The rank vector is a solution of $(A - I)\mathbf{x} = \mathbf{0}$.

$$A - I = \begin{bmatrix} -1 & 0 & \frac{1}{2} \\ 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ t \\ t \end{bmatrix},$$

So the rank vector can be $(\frac{1}{2}, 1, 1)$ (showing that the ranks of v_2 and v_3 are equal, and higher than the rank of v_1).

2. *Find the eigenvalues and eigenvectors of the matrix below. Using these, find the matrices P and D such that $A = PDP^{-1}$, where D is a diagonal matrix. (You do not have to calculate P^{-1}.)* *(6 marks)*

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ -1 & 0 & -2 \end{bmatrix}$$

Solution:

$$|A - xI| = \begin{vmatrix} 1-x & 0 & 2 \\ 1 & -x & 2 \\ -1 & 0 & -2-x \end{vmatrix} = -x \begin{vmatrix} 1-x & 2 \\ -1 & -2-x \end{vmatrix} = -x(x^2 + x) = -x^2(x+1),$$

so the eigenvalues are $0, 0, -1$. The eigenvectors for $\lambda = 0$:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ -1 & 0 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{If } A\mathbf{x} = \mathbf{0} \text{ then } \begin{cases} x_1 = -2t \\ x_2 = s \\ x_3 = t \end{cases} \Rightarrow \mathbf{x} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvectors for $\lambda = -1$:

$$A + I = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{If } (A+I)\mathbf{x} = \mathbf{0} \text{ then } \begin{cases} x_1 = -t \\ x_2 = -t \\ x_3 = t \end{cases} \Rightarrow \mathbf{x} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{So } P^{-1}AP = D, \text{ that is, } A = PDP^{-1} \text{ for } P = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(Be careful! The eigenvectors should be in the same order in P as their eigenvalues in D.)

3. *Determine whether the following matrices are self-adjoint, unitary or normal (this means 6 answers!). Justify each of your answers.* *(5 marks)*

$$A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Solution: $A^* = \overline{A^T} = \overline{\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}} = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} = A$, so A is self-adjont, and thus it is also normal. A square matrix is unitary if and only if their columns are orthonormal. But the absolute value of the first column of A is $\sqrt{3} \neq 1$, so A cannot be unitary.

B is a real matrix, so $B^* = B^T$. $B^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \neq B$ and the first column of B has absolute value $\sqrt{2} \neq 1$ (or: the first and the third columns of B are not orthogonal), so B is not self-adjoint, and not unitary. Hence we must check normality.

$$B^TB = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix} \text{ and } BB^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \text{ so } B^TB \neq BB^T \;\Rightarrow B \text{ is not normal.}$$

4. *Orthogonalize the column vectors of the following matrix A by Gram–Schmidt's method. Use this to give the (reduced) QR decomposition of A.* *(5 marks)*

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Solution: For $\mathbf{b}_1 = (1, -1, 1, 1)$ and $\mathbf{b}_2 = (3, 1, -1, 3)$,
$\mathbf{c}_1 = (1, -1, 1, 1)$, and $\mathbf{c}_2 = \mathbf{b}_2 - \frac{\langle \mathbf{c}_1, \mathbf{b}_2 \rangle}{|\mathbf{c}_1|^2}\mathbf{c}_1 = (3, 1, -1, 3) - \frac{4}{4}(1, -1, 1, 1) = (2, 2, -2, 2)$ or we can take $\tilde{\mathbf{c}}_2 = (1, 1, -1, 1)$ instead.
From this, the corresponding orthonormal system is $\mathbf{q}_1 = \frac{1}{2}(1, -1, 1, 1)$ and $\mathbf{q}_2 = \frac{1}{2}(1, 1, -1, 1)$, so

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } R = Q^TA = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix},$$

for which $QR = A$.