**Vector and Matrix Algebra** **Problem sheet 10** **Autumn 2023**

**1.** Find the nonzero terms in the big formula of $\det(A)$ (out of the 24). What are their signs?

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 0 \\ 5 & 6 & 0 & 7 \\ 8 & 0 & 0 & 9 \end{bmatrix}$$

**2.** True or false?

(i) If the main diagonal entries of a matrix are all 0 then the determinant is 0.

(ii) If $|A| = 0 = |B|$ then $|A+B| = 0$.

(iii) If $|A| = 0 = |B|$ then $|AB| = 0$.

(iv) If $|A| = 0$ then there is an entry of $A$ which is 0.

(v) If $|A| > 0$ then there is an entry of $A$ which is positive.

**3.** Write recursive formulas for $a_n = |A_n|$, $b_n = |B_n|$ and $c_n = |C_n|$ and determine their values.

$$A_n = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}; \; B_n = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}; \; C_n = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 1 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 1 & -1 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix}.$$

**4.** Find all the cofactors of $A_3$ of the previous exercise. Form the cofactor matrix, the adjugate matrix and $A_3^{-1}$.

**5.** Use Cramer's Rule to solve $\begin{matrix} x+ & 2y & = 1 \\ 2x+ & y & = 3 \end{matrix}$.

**6.** a) Determine the volume of the tetrahedron (inscribed into the unit cube) with vertices: $(0;0;0)$, $(1;1;0)$, $(1;0;1)$, $(0;1;1)$. b)**HW** Let $P = (2;1)$, $Q = (7;2)$, $R = (-1;8)$ and $S = (0;9)$. Determine the area of the triangle $PQR$ and of the quadrangle $PQSR$.

**7.** Argue that the area of a triangle that has integer lattice point coordinates is an integer multiple of $1/2$ and the volume of a tetrahedron that has integer lattice point coordinates is an integer multiple of $1/6$.

**8.** Suppose $A$ has orthonormal columns. Using $A^T A$ justify that $\det(A) = \pm 1$.

**9.** Rearranging the three vectors, which of the triple products $( \quad \times \quad ) \cdot \quad$ are the same as $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$?

**10.** A point $\mathbf{u}$ in space is the position of the mass on which a force $\mathbf{F}$ acts (also a 3-dimensional vector). Observe that if $\mathbf{u}$ and $\mathbf{F}$ are parallel then there is no turning. However if they are not parallel then there is. The rotational force, the torque is exactly $\mathbf{u} \times \mathbf{F}$. Check that the direction is correct.

**11.** Given a matrix your friend picks a row. You win if you can change at most one element of that row to make the determinant 0. Can you always win?

**12.** Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$. Determine the eigenvalues and corresponding eigenvectors of $A$ and of $A+I$.

**13.** Determine the eigenvalues and corresponding eigenvectors of $B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, of $B^2$ and of $B^{-1}$.

**14.** Explain the general phenomena governing the previous two exercises.