1. Determine the eigenvalues and the singular values of A and B, where $\varepsilon = 0.001$. Comment on how do they change when we alter the lower left corner by the tiny ε?

$$A = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 10 \\ \varepsilon & 0 \end{bmatrix}.$$

2. Determine the matrix for each of the following linear transformations:

(i) reflection through the x axis;

(ii) reflection through the origin;

(iii) rotation by 60° (in positive direction);

(iv) orthogonal projection onto the line $x = y$.

3. True or false?

(i) If $A = (a_{ij})_{i,j}$ then $\mathrm{tr}(A^T A) = \sum_{i,j} a_{ij}^2$.

(ii) If A is 2×2 and has two eigenvalues ± 1 then its singular values are $\sigma_1 = \sigma_2 = 1$.

(iii) If A has no 0 singular value then AA^T or $A^T A$ is invertible.

(iv) If A is symmetric then its singular values are equal to the absolute values of its eigenvalues.

(v) If A is orthogonal then its singular values are all 1.

4. Find the SVD of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Simon says: "If $A^T A$ is diagonal then it is easy to obtain the SVD!" Really, how?

6. HW Determine the reduced SVD and the pseudoinverse of

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -2 \end{bmatrix}.$$

7. Suppose A has full column rank. Explain why $A^T A$ is invertible. Show that $B = (A^T A)^{-1} A^T$ is a left inverse of A, that is $BA = I$. Explain this in terms of the four subspaces. Explain also that $B = A^+$.

8. Based on the previous exercise what is the pseudoinverse of A if A has full row rank?

9. Find the pseudoinverse of the following 3 matrices using the previous two exercises or/and the SVD method.

$$A = \begin{bmatrix} 1 \\ -3 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 & 2 \end{bmatrix}; \qquad C = AB = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}.$$

Confirm that $(AB)^+ = B^+ A^+$. (This is always true if A is of maximal column rank and B is of maximal row rank. Such factorisation is called "full rank factorisation.")

10. Let T be the linear transformation of $\mathbb{R}^3$ defined by circular motion: $T((a, b, c)) = (b, c, a)$. What is $T^2(\mathbf{v}) = T(T(\mathbf{v}))$? What is T^3 and T^{100} doing on $\mathbb{R}^3$?

11. (Hard) Given $Ax = b$, 3 equations in 2 variables, is like giving a triangle on the plane. Describe the point $A^+ b$ (the minimum norm least square solution) in terms of the geometry of the triangle. (You can assume that in each equation the sumsquares of the coefficients is 1.)