Vector and Matrix Algebra **Problem sheet 2** **Autumn 2023**

1. Let $\mathbf{v}, \mathbf{w}$ be two independent vectors on the plane or in higher dimension. What is the set $L = \{\lambda\mathbf{v} + \mu\mathbf{w} \mid \lambda, \mu \text{ integers}\}$? What is the set $K = \{\lambda\mathbf{v} + \mu\mathbf{w} \mid \lambda, \mu > 0\}$?

2. True or false?

(i) When solving a system of linear equations pivots are always positive.

(ii) If a system has two solutions then it has more.

(iii) The triangle inequality is always strict: $||\mathbf{v}|| + ||\mathbf{w}|| > ||\mathbf{v} - \mathbf{w}||$.

3. Multiply A times $\mathbf{x}$ to find the components of $A\mathbf{x}$ in each case:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What should be $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ is solvable?

4. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ and a real parameter t we have $||(\mathbf{a} - t\mathbf{b})|| \geq 0$. Take the square, decompose and you obtain an inequality, quadratic in t. Using the discriminant method deduce the Cauchy-Schwarz-Bunyakovsky inequality.

5. Consider the following system of linear equations:

$$\begin{aligned} x + \ \ y + \ \ z &= 2 \\ x + 2y + \ \ z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

Interpret the planes and their intersection lines. Describe all the solutions of the system.

6. Consider the following two systems of linear equations (c a real parameter):

$$\begin{aligned} x - \ \ y + \ \ z &= 1 \\ x - 2y - \ \ z &= -2 \\ 2x - 3y + cz &= 1. \end{aligned} \qquad \begin{aligned} x - \ \ y + z &= 1 \\ 2x - 2y - z &= -1 \\ 2x + cy + z &= 3. \end{aligned}$$

How does the number of solutions depend on c in each system?

7. **HW** Apply elimination with circling the pivots of the system. Name and describe the elimination matrices!

$$\begin{aligned} 2x - 3y \qquad &= 3 \\ 4x - 5y + \ \ z &= 7 \\ 2x - \ \ y - 3z &= 5. \end{aligned}$$

Solve by back(ward) substitution.

8. Construct a 3×3 system that needs two row exchanges to reach a triangular form.

9. Suppose the first two columns are the same. What happens during elimination?

10. For which numbers a does elimination stop before reaching a full triangle in

$$\begin{bmatrix} a & 1 \\ a & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}?$$