1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$. Show that the first three columns are independent, but all four are dependent. Describe the dependencies using the equation $A\mathbf{v} = \mathbf{0}$. Show that the four columns of A span $\mathbb{R}^3$. Do they form a basis? Which columns form a basis?

2. True or false?

(i) A and A^T have the same number of pivots.

(ii) A and A^T have the same left nullspace.

(iii) If the row space of A equals the column space of A then $A = A^T$.

(iv) If $A = -A^T$ then the row space of A equals the column space of A.

(v) The columns of A fom a basis of $C(A)$.

3. Which of the columns form a basis of $C(R)$? Describe the basis of $N(R)$. Answer these questions for R^T, too.

$$R = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Find the complete solution of the following systems of equations.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 3 & -1 & 1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

5. Let $M = M^{3\times 3}$, the vector space of real 3×3 matrices.

(i) Let W denote the subspace of M spanned by diagonal matrices. What is W? What is $\dim(W)$?

(ii) Let V denote the subspace of M spanned by rank-1 matrices. What is V? What is $\dim(V)$?

6. Construct many independent 3×3 matrices of row echelon form. How many could you find?

7. You are allowed to put four 1's into a 3×3 matrix, the rest of the entries are 0. How to do this if you want to keep the dimension of the column space as small as possible? How to do this if you want to keep the dimension of the row space as small as possible? How to do this if you want to keep the dimension of the nullspace as small as possible? What is the minimum of the sum of the dimensions of all the Four Fundamental Subspaces?

8. Construct a matrix with the following property or refute the possibility.

(i) $C(A)$ contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $C(A^T)$ contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

(ii) $C(A)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and $N(A)$ has basis $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(iii) **(HW)** $\dim N(A) = 1 + \dim N(A^T)$.

(iv) **(HW)** $N(A)$ contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $C(A)$ contains $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(v) **(HW)** $C(A) = C(A^T)$ but $N(A) \neq N(A^T)$.

9. Verify that $C(AB) \leq C(A)$ and conclude that $r(AB) \leq r(A)$. Show similarly that $r(AB) \leq r(B)$.