1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \end{bmatrix}$. Determine bases for each of $C(A), C(A^T), N(A), N(A^T)$. Check the orthogonality of the respective subspaces.

2. True or false?

(i) A and A^T are orthogonal to each other.

(ii) $N(A)$ and $N(A^T)$ are orthogonal complements.

(iii) If the row space of A equals the column space of A then $N(A) = N(A^T)$.

(iv) The projection on $N(A^T)$ is $I - A(A^TA)^{-1}A^T$ if A has independent columns.

(v) If $C(A)$ and $N(A)$ are orthogonal complements then $A = A^T$.

3. The following system of equations has no solution:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 1 & 3 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} \qquad A\mathbf{x} = \mathbf{b}.$$

Find a combination of the equations that produces $0 = 1$. Reinterpret this to finding a vector $\mathbf{y} \in N(A^T)$ such that $\mathbf{y}^T\mathbf{b} = 1$.
Prove that given an $m \times n$ matrix A and vector $\mathbf{b} \in \mathbb{R}^m$ then either there exists $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ or there exists $\mathbf{y} \in N(A^T)$ such that $\mathbf{y}^T\mathbf{b} = 1$.

4. Determine the projection matrix onto the column space and onto the left nullspace of R_1. Do the same for R_2. Comment on the results.

$$R_1 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}; \qquad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

5. Let $W \leq V = \mathbb{R}^{10}$ have dimension 7.

(i) What is the dimension of its orthogonal complement $W^\perp$?

(ii) What are the possible dimensions of subspaces $U \leq V$ orthogonal to W?

(iii) Suppose $C(A) = W$. What are the minimal number of columns/rows of A?

(iv) Suppose $N(A) = W$. What are the minimal number of columns/rows of A?

6. HW Let $x + y - 2z = 0$ describe a plane $\mathcal{P}$ in the space $\mathbb{R}^3$. Determine the 1×3 matix A for which $N(A) = \mathcal{P}$. Find the special solutions s_1, s_2 and a basis for the orthogonal complement $\mathcal{P}^\perp$. Split $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ into nullspace and row space components.

7. Determine the projection matrices onto the column space of R and onto the left nullspace. Answer these questions for R^T, too.

$$R = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$