1. Let $\mathbf{b} \in \mathbb{R}^m$ and all entries of $\mathbf{a} \in \mathbb{R}^m$ are 1. We consider $\mathbf{a}x = \mathbf{b}$ with unique real variable x. Let $\hat{x}$ denote the least squares solution. Then

(i) $\mathbf{a}^T\mathbf{a}\hat{x} = \mathbf{a}^T\mathbf{b}$ shows that $\hat{x}$ is the average (mean) of the entries of $\mathbf{b}$;

(ii) $||\mathbf{b} - \mathbf{a}\hat{x}||^2$ is the variance of the entries of $\mathbf{b}$ (its square root is the standard deviation);

(iii) find the projection matrix P onto the line through $\mathbf{a}$.

2. True or false?

(i) If Q is an orthogonal matrix then so is Q^{-1}.

(ii) If an $m \times n$ matrix A has orthogonal columns then $||A\mathbf{v}|| = ||\mathbf{v}||$ for every vector $\mathbf{v} \in \mathbb{R}^n$.

(iii) A has three orthogonal columns of length $1, 2$ and 3 if and only if A^TA is a diagonal matrix with diagonal entries $1, 4$ and 9.

(iv) If an orthogonal matrix is lower triangular then it is diagonal.

(v) The determinant of $I + A$ is $1 + |A|$.

3. Let $b_1 = 0$, $b_2 = 8$, $b_3 = 8$, $b_4 = 20$ be the measurements at times $t_1 = 0$, $t_2 = 1$, $t_3 = 3$, $t_4 = 4$. Find the best straight line approximating these points on the plane, by determining and solving the normal equation $A^TA\widehat{\mathbf{x}} = A^T\mathbf{b}$. What is the minimum (total) error?

4. Let $\mathbf{e}$ be the error vector for a projection problem. Which is $\mathbf{e}$ perpendicular to: $\mathbf{b}$, $\widehat{\mathbf{x}}$, $\mathbf{e}$, $\mathbf{p}$? Show that $||\mathbf{e}||^2 = \mathbf{e}^T\mathbf{b} = \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\mathbf{p}$.

5. Perform the Gram-Schmidt orthogonalisation algorithm for the vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Do the same for the columns of $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

6. HW Let $x + y - 2z = 0$ describe a plane $\mathcal{P}$ in the space $\mathbb{R}^3$. Find two orthogonal vectors in $\mathcal{P}$ and make them orthonormal. Extend it to an orthonormal basis of $\mathbb{R}^3$.

7. Determine the following determinants:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{vmatrix}, \qquad \begin{vmatrix} 1 & a & a^2 \\ a & 1 & a \\ a^2 & a & 1 \end{vmatrix}, \qquad \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}, \qquad \det\left([1\ 2\ 3] \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right).$$

8. Find the determinant of rotations and reflections:

$$Q_1 = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} \cos 2\vartheta & -\sin 2\vartheta \\ -\sin 2\vartheta & -\cos 2\vartheta \end{bmatrix}.$$

9. Observe that 7 divides 343, 147 and 504. Surprise surprise, 7 divides the following determinant. Why?

$$\begin{vmatrix} 3 & 4 & 3 \\ 1 & 4 & 7 \\ 5 & 0 & 4 \end{vmatrix}.$$

10. What is the determinant of a projection matrix?