1. *True of false?*

(i) Vectors are dependent if their dot product is 0.

(ii) If the linear combination of $\mathbf{v}$ *and* $\mathbf{w}$ *form a plain then* $\mathbf{v}$ *and* $\mathbf{w}$ *are independent.*

(iii) $2\mathbf{v}$ *is always longer than* $\mathbf{v}$.

(iv) $\mathbf{a}-\mathbf{b}$ *is the same as* $\mathbf{a}+(-\mathbf{b})$.

(v) $\mathbf{v}$ *and* $\mathbf{w}$ *are on a common line through the origin (called collinear) if and only if* $\frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{\mathbf{w}}{\|\mathbf{w}\|}$.

(vi) Any vector is a matrix.

Solution: (i) No, actually they are almost always independent. They are dependent only when at least one of them is $\mathbf{0}$.

(ii) Yes.

(iii) No: for $\mathbf{v}=\mathbf{0}$ we have $\|\mathbf{v}\|=\|2\mathbf{v}\|=0$. But it is true if $\mathbf{v}\neq 0$.

(iv) Yes.

(v) No, these vectors can also be the opposites of each other.

(vi) Yes, a vector in $\mathbb{R}^n$ is an $n\times 1$ matrix.

2. *Let*

$$\mathbf{a}=\begin{bmatrix}1\\0\\-1\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}1\\1\\0\end{bmatrix},\quad \mathbf{c}=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad \mathbf{d}=\begin{bmatrix}1\\2\\1\end{bmatrix}.$$

Describe algebraically and geometrically the linear combinations of $\mathbf{b}$ *and* $\mathbf{c}$. *Describe also the linear combinations of* $\mathbf{a}$ *and* $\mathbf{b}$. *Which of* $\mathbf{c},\mathbf{d}$ *can be expressed using* $\mathbf{a}$ *and* $\mathbf{b}$? *Determine the subsets of* $\{\,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\,\}$ *that are independent.*

Solution: There no parallel vectors among the four, so any two of them spans (that is, gives as the set of linear combinations) a plane containing the origin. In particular, the linear combinations of $\mathbf{b}$ and $\mathbf{c}$ are

$$\left\{x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}0\\0\\1\end{bmatrix}\;\middle|\; x,y\in\mathbb{R}\right\}=\left\{\begin{bmatrix}x\\x\\y\end{bmatrix}\;\middle|\; x,y\in\mathbb{R}\right\},$$

and those of $\mathbf{a}$ and $\mathbf{b}$ are

$$P:=\left\{x\begin{bmatrix}1\\0\\-1\end{bmatrix}+y\begin{bmatrix}1\\1\\0\end{bmatrix}\;\middle|\; x,y\in\mathbb{R}\right\}=\left\{\begin{bmatrix}x+y\\y\\-x\end{bmatrix}\;\middle|\; x,y\in\mathbb{R}\right\}.$$

$\mathbf{c}\notin P$, since that would give $x+y=0$, $y=0$, $-x=1$, which is contradictory ($(-1,0)$ is not a solution of the first equation) but for $\mathbf{d}$ we get the equations $x+y=1$, $y=2$, $-x=1$, and it has a solution: $x=-1$, $y=2$. So $\mathbf{d}=-\mathbf{a}+2\mathbf{b}$.

The one-element subsets of $\{\,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\,\}$ are independent because there are no zero vectors among them. The two-element subsets are also independent, since neither of the vectors is parallel to another (so any two of them span a plane). The three-element subsets are independent if and only if they span the whole space, that is, two of its vectors span a plane which does not contain the third. Since $\mathbf{c}\notin P$ but $\mathbf{d}\in P$, the set $\{\,\mathbf{a},\mathbf{b},\mathbf{c}\,\}$ is independent but $\{\,\mathbf{a},\mathbf{b},\mathbf{d}\,\}$ is not. For the other two three-element subsets, let us consider the plane spanned by $\mathbf{c}$ and $\mathbf{d}$: $\{(y,2y,x+y)\mid x,y\in\mathbb{R}\,\}$. Here the second entry is always the double of the first, so neither $\mathbf{a}$ nor $\mathbf{b}$ is in this plane. Thus the other three-element subsets, $\{\,\mathbf{a},\mathbf{c},\mathbf{d}\,\}$ and $\{\,\mathbf{b},\mathbf{c},\mathbf{d}\,\}$ are also independent. Finally, $\{\,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\,\}$ is not independent, since $\mathbf{a}-2\mathbf{b}+\mathbf{d}=\mathbf{0}$ is a nontrivial linear combination giving $\mathbf{0}$.

3. *Let a clock have unit radius and let* $\mathbf{a}_1,\dots,\mathbf{a}_{12}$ *denote the twelve vectors pointing from the centre of the clock (the origin) to the twelve round hours. Determine the entries* $\mathbf{a}_1$, $\mathbf{a}_4$ *and* $\mathbf{a}_9$. *What is the sum* $\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_{12}$? *What is the sum* $\mathbf{a}_1+\mathbf{a}_5+\mathbf{a}_9$?

Solution: $\mathbf{a}_1=(\cos 60°,\sin 60°)=(\frac{1}{2},\frac{\sqrt{3}}{2})$, $\mathbf{a}_4=(\cos(-30°),\sin(-30°))=(\frac{\sqrt{3}}{2},-\frac{1}{2})$ and $\mathbf{a}_9=(-1,0)$.

Since in the sum $\mathbf{a}_1+\ldots+\mathbf{a}_{12}$ every vector has its opposite ($\mathbf{a}_1$ and $\mathbf{a}_7$, $\mathbf{a}_2$ and $\mathbf{a}_8$, …, $\mathbf{a}_6$ and $\mathbf{a}_{12}$

are the negatives of each other), the sum must be $\mathbf{0}$.
We can calculate the sum $\mathbf{a}_1+\mathbf{a}_5+\mathbf{a}_9$ by the coordinates: $\mathbf{a}_5 = (\cos(-60°), \sin(-60°)) = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$, so

$$\mathbf{a}_1+\mathbf{a}_5+\mathbf{a}_9 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or notice that $\mathbf{a}_1+\mathbf{a}_5 = \mathbf{a}_3 = -\mathbf{a}_9$ (since $\mathbf{a}_1$ and $\mathbf{a}_5$ are two sides of a parallelogram whose diagonal is $\mathbf{a}_3$), so $\mathbf{a}_1+\mathbf{a}_5+\mathbf{a}_9 = \mathbf{0}$.

4. *Find two nonzero vectors* $\mathbf{a}, \mathbf{b}$ *that are perpendicular to* $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *and to each other!* **HW** *with* $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Solution: $\mathbf{a}$ and $\mathbf{b}$ must be two solutions of the equation

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + y = 0,$$

which are also perpendicular to each other. It is easy to find one nonzero solution, for example $\mathbf{a} = (1, -1, 0)$. Then we want $\mathbf{b}$ to be perpendicular to both vectors, that is, $\mathbf{b} = (x, y, z)$ must satisfy the equations

$$x + y = 0 \text{ and } \mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x - y = 0,$$

so $x = y = 0$, but z can be arbitrary, say $\mathbf{b} = (0, 0, 1)$.
(Another way to find the a third vector would be to take the cross product of the first two)

5. *Show these properties of the dot product:*
(i) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;
(ii) $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$;
(ii) $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$;

Solution: (i) $\mathbf{v} \cdot \mathbf{w} = \sum_i v_i w_i = \sum_i w_i v_i = \mathbf{w} \cdot \mathbf{v}$.
(ii) $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = (v_1, \dots, v_n)(w_1 + u_1, \dots, w_n + u_n) = \sum_i v_i(w_i + u_i) = \sum_i (v_i w_i + v_i u_i) = (\sum_i v_i w_i) + (\sum_i v_i u_i) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$.
(iii) $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$, where we used the already proved part (i), (ii), and again (i).

6. *Let* $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. *Compute the real number(s) c for which* $\mathbf{w} - c\mathbf{v}$ *is perpendicular to* $\mathbf{v}$. *Can you do it for arbitary* $\mathbf{v}$ *and* $\mathbf{w}$*?*

Solution:

$$\mathbf{w} - c\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - c \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-c \\ 2-c \end{bmatrix}, \text{ and we need } \mathbf{v} \cdot (\mathbf{w} - c\mathbf{v}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1-c \\ 2-c \end{bmatrix} = 3 - 2c = 0.$$

So the solution is $c = \frac{3}{2}$. Indeed, $\mathbf{w} - \frac{3}{2}\mathbf{v} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This way we can write $\mathbf{w}$ as the sum of two perpendicular vectors, one parallel with $\mathbf{v}$ and one perpendicular to $\mathbf{v}$: $\mathbf{w} = c\mathbf{v} + (\mathbf{w} - c\mathbf{v}) = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$.

If $\mathbf{v} \neq \mathbf{0}$ then we can also decompose any vector $\mathbf{w}$ (even in the n dimensional space) into the sum $\mathbf{w} = \mathbf{w}' + \mathbf{w}''$, where $\mathbf{w}' || \mathbf{v}$ and $\mathbf{w}'' \perp \mathbf{v}$. Let $\mathbf{w}' = c\mathbf{v}$, and then $\mathbf{w}'' = \mathbf{w} - c\mathbf{v}$. For this we need $\mathbf{v} \cdot (\mathbf{w} - c\mathbf{v}) = \mathbf{v} \cdot \mathbf{w} - c(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot \mathbf{w} - c\|\mathbf{v}\|^2 = 0$, that is, $c = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}$. So

$$\mathbf{w} = \mathbf{w}' + \mathbf{w}'', \text{ where } \mathbf{w}' = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v} \text{ and } \mathbf{w}'' = \mathbf{w} - \mathbf{w}'$$

is the decomposition we were looking for. This $\mathbf{w}'$ is the *orthogonal projection* of $\mathbf{w}$ to the vector $\mathbf{v}$, and the solution above shows that it is unique.

7. *Using Pythagoras' Theorem, prove the triangle inequality* $\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\|+\|\mathbf{w}\|$.

Solution: Let's follow the intuitive notion that the shortest route from a point to another is along a straight line. That is, instead of going along the vector $\mathbf{v}$ and then continue from that point along the vector $\mathbf{w}$, we may travel on their orthogonal projections $\mathbf{v}'$ and $\mathbf{w}'$ on the vector $\mathbf{u} := \mathbf{v}+\mathbf{w}$. It follows from the theorem of Pythagoras that the length of the projection is not longer than the original vector: $\|\mathbf{v}\|^2 = \|\mathbf{v}'+\mathbf{v}''\|^2 = \|\mathbf{v}'\|^2+\|\mathbf{v}''\|^2 \geq \|\mathbf{v}'\|^2$, since $\mathbf{v}' \perp \mathbf{v}''$.
On the other hand, the uniqueness of the projection implies $\mathbf{v}+\mathbf{w} = \mathbf{v}'+\mathbf{w}'$ because $\mathbf{u} = (\mathbf{v}+\mathbf{u})+\mathbf{0} = (\mathbf{v}'+\mathbf{w}')+(\mathbf{v}''+\mathbf{w}'')$ are both sums of a vector parallel to $\mathbf{u}$ and a vector perpendicular to $\mathbf{u}$.
Finally, if $\mathbf{v}' = c\mathbf{u}$ and $\mathbf{w}' = d\mathbf{u}$, then $\|\mathbf{v}'+\mathbf{w}'\| = \|(c+d)\mathbf{u}\| = |c+d|\|\mathbf{u}\| \leq (|c|+|d|)\|\mathbf{u}\| = |c|\|\mathbf{u}\|+|d|\|\mathbf{u}\| = \|c\mathbf{u}\|+\|d\mathbf{u}\| = \|\mathbf{v}'\|+\|\mathbf{w}'\|$.
So $\|\mathbf{v}+\mathbf{w}\| = \|\mathbf{v}'+\mathbf{w}'\| \leq \|\mathbf{v}'\|+\|\mathbf{w}'\| \leq \|\mathbf{v}\|+\|\mathbf{w}\|$.

8. *How many vectors can you find on the plane/space such that every pair has negative dot product?*

Solution: First of all, let us notice that the dot product of two vectors is negative if and only if their angle is greater than 90° (it follows, for instance, from the formula in the solution of problem 6 that the dot product is negative if and only if the projection of the vector $\mathbf{w}$ on $\mathbf{v}$ points in the opposite direction as $\mathbf{v}$, that is, the angle of $\mathbf{v}$ and $\mathbf{w}$ is $> 90°$).
In the plane there can be at most three such vectors, since the angle between the neighbouring ones are $> 90°$, and the sum of the angles is 360°. And, clearly, there exist such three vectors, for example the vectors pointing from the center of a regular triangle to its vertices.
If $\mathbf{n}, \mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors in $\mathbb{R}^3$ so that the angle between any two of them is $> 90°$, then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are all in the same half space opposite to $\mathbf{n}$, and we can reduce the question to the planar case by taking the projections of $\mathbf{v}_i$-s to the separating plane.
So let P be the plane perpendicular to $\mathbf{n}$ (as in problem 9), the projection of $\mathbf{v}_i$ on $\mathbf{n}$ the vector $c_i\mathbf{n}$, where $c_i < 0$, since $\mathbf{v}_i \cdot \mathbf{n} < 0$, finally $\mathbf{v}_i' = \mathbf{v}_i - c_i\mathbf{n}$ for $i = 1, \ldots, k$ (then $\mathbf{v}_i'$ is the projection of $\mathbf{v}_i$ on the plane P). Then for $i \neq j$ $0 > \mathbf{v}_i \cdot \mathbf{v}_j = (\mathbf{v}_i'+c_i\mathbf{n}) \cdot (\mathbf{v}_j'+c_j\mathbf{n}) = \mathbf{v}_i' \cdot \mathbf{v}_j' + c_ic_j\|\mathbf{n}\|^2 \geq \mathbf{v}_i' \cdot \mathbf{v}_j'$, since $\mathbf{v}_1', \ldots, \mathbf{v}_k'$ are perpendicular to $\mathbf{n}$, and $c_ic_j > 0$. So $\mathbf{v}_1', \ldots, \mathbf{v}_k'$ are vectors of a plane whose pairwise angles are $> 90°$, so by the planar case, $k \leq 3$.
This shows that there are at most 4 such vectors in $\mathbb{R}^3$, and indeed, there exist four, for example the vectors pointing from the center of a regular tetrahedron to its corners.

9. *To every space vector $\mathbf{v} \neq \mathbf{0}$ there corresponds a plane $P_\mathbf{v}$ consisting of the vectors perpendicular to $\mathbf{v}$: $P_\mathbf{v} = \{\, \mathbf{w} \mid \mathbf{w} \cdot \mathbf{v} = 0 \,\}$. Describe the possible configurations of the planes corresponding to three independent vectors. Similarly, describe the possible configurations of the planes corresponding to three dependent vectors.*

Solution: The intersection of the planes contains exactly those vectors $\mathbf{v}$ that are perpendicular to to all of the three vectors, so $\mathbf{v}$ is perpendicular to any linear combination of these vectors. If the three vectors are independent then they span the whole space, so $\mathbf{v}$ is perpendicular to any vector of $\mathbb{R}^3$, including itself: $0 = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$, which implies that $\mathbf{v} = \mathbf{0}$. So these planes intersect only in the origin.
If the three vectors are dependent but two of them are independent, then the planes corresponding to these two intersect in a line, and the third vector is a linear combination of these two, so every vector in this line is also perpendicular to the third. So in this case the three planes intersect in a line.
If all three vectors are parallel then they determine the same plane.

10. *To every plane P (not necessarily containing the origin!) in the space there corresponds a (nonunique) vector $\mathbf{v}_P \neq \mathbf{0}$ perpendicular to P: $\mathbf{v}_P \cdot (\mathbf{w}_1 - \mathbf{w}_2) = 0$ for all $\mathbf{w}_1, \mathbf{w}_2 \in P$. Describe the configurations of three planes when the corresponding three vectors are independent and when they are dependent.*

Solution: We call such a vector $\mathbf{v}_P$ a normal vector of the plane P. If the normal vectors of two planes are independent then the two planes intersect in one line.
If the normal vectors of three planes are independent then the intersection lines of the pairs of planes cannot be parallel because otherwise all of the three normal vectors would be perpendicular

to these lines, so the normal vectors would be in one plane, that is, they would be dependent. But any two of these intersection lines are in one plane so the nonparallel ones intersect in one point. This point is the intersection of the three planes.
If two normal vectors are independent but the third is a linear combination of these two then the intersection line of the first two is parallel with the third plane. So the intersection of the three planes is either empty or it is a line.
If the three normal vectors are pairwise parallel then the three planes are also parallel, so their intersection is empty unless all of them coincides.