1. *Let $\mathbf{v}, \mathbf{w}$ be two independent vectors on the plane or in higher dimension. What is the set $L = \{\lambda\mathbf{v} + \mu\mathbf{w} \mid \lambda, \mu$ integers$\}$? What is the set $K = \{\lambda\mathbf{v} + \mu\mathbf{w} \mid \lambda, \mu > 0\}$?*

Solution: L consists of the crosspoints of an infinite grid of congruent parallelograms, one of which is the parallelogram defined by the vectors $\mathbf{v}$ and $\mathbf{w}$. K is the smaller of the two regions of the plane whose borders are the halflines defined by the two vectors. (Think about what we get for K and L if the two vectors are dependent.)

2. *True of false?*
(i) When solving a system of linear equations pivots are always positive.
(ii) If a system has two solutions then it has more.
(iii) The triangle inequality is always strict: $\|\mathbf{v}\| + \|\mathbf{w}\| > \|\mathbf{v} - \mathbf{w}\|$.

Solution: (i) False.
(ii) True: if we have two solutions $\mathbf{x}$ and $\mathbf{y}$ then substituting their difference $\mathbf{y} - \mathbf{x}$, or any scalar multiple of this, we get $\mathbf{0}$, so $\mathbf{x} + t(\mathbf{y} - \mathbf{x})$ $(t \in \mathbb{R})$ gives infinitely many different solutions.
(iii) False. It is an equality if $\mathbf{v}$ and $\mathbf{w}$ are parallel, and point in the opposite direction. For example for $\mathbf{v} = (3, 4)$ and $\mathbf{w} = (-6, -8)$ we have $\|\mathbf{v}\|+\|\mathbf{w}\| = 5+10 = 15$, and $\|\mathbf{v}-\mathbf{w}\| = \|(9, 12)\| = 15$. (But a more obvious counterexample is $\mathbf{v} = \mathbf{w} = \mathbf{0}$.)

3. *Multiply A times $\mathbf{x}$ to find the components of $A\mathbf{x}$ in each case:*

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What should be $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ is solvable.

Solution:

$$\begin{bmatrix} z \\ y \\ x \end{bmatrix} \qquad \begin{bmatrix} x + y + z \\ 2x + 2y + 2z \\ 0 \end{bmatrix} \qquad \begin{bmatrix} y \\ 2x + y \\ x \end{bmatrix}$$

Clearly, the first is always solvable (for $\mathbf{b} = (b_1, b_2, b_3)$, $(x, y, z) = (b_3, b_2, b_1)$. In the second b_3 must be zero, and $b_2 = 2b_1$, to make the equation solvable (but then we have infinitely many different solutions). In the third the solution can only be $x = b_3$ and $y = b_1$, but it also has to satisfy $2x + y = b_2$, so this equation is solvable only when $b_2 = 2b_3 + b_1$.

4. *Given two vectors $\mathbf{a}$ and $\mathbf{b}$ and a real parameter t we have $\|(\mathbf{a} - t\mathbf{b})\| \geq 0$. Take the square, decompose and you obtain an inequality, quadratic in t. Using the discriminant method deduce the Cauchy–Schwarz–Bunyakovsky inequality.*

Solution: $\|(\mathbf{a} - t\mathbf{b})\|^2 = (\mathbf{a} - t\mathbf{b}) \cdot (\mathbf{a} - t\mathbf{b}) = t^2\|\mathbf{b}\|^2 - 2t\mathbf{a} \cdot \mathbf{b} + \|\mathbf{a}\|^2 \geq 0$ for every t. So the corresponding quadratic equation, $\|(\mathbf{a} - t\mathbf{b})\|^2 = 0$, has at most one solution in t if $\mathbf{b} \neq 0$, and only when $\mathbf{a}$ and $\mathbf{b}$ are dependent: the t for which $\mathbf{a} = t\mathbf{b}$. So the discriminant as a quadratic equation in t is ≤ 0. This means that

$$4(\mathbf{ab})^2 - 4\|\mathbf{a}\|^2\|\mathbf{b}\|^2 \leq 0,$$

so $(\mathbf{ab})^2 \leq \|\mathbf{a}\|^2\|\mathbf{b}\|^2$, and taking the square root of both sides we get

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|.$$

An alternative proof can be given by using the projection of a vector to the other. If $\mathbf{b} \neq \mathbf{0}$ then the orthogonal projection of $\mathbf{a}$ on $\mathbf{b}$ is $\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$ (see problem 1/6.), and its length is at most $\|\mathbf{a}\|$ by the theorem of Pythagoras, so

$$\left\| \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \right\| = \left| \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right| \|\mathbf{b}\| \leq \|\mathbf{a}\|,$$

which easily gives the inequality to be proved. On the other hand, if $\mathbf{b} = \mathbf{0}$, then $|\mathbf{a} \cdot \mathbf{b}| = 0 = \|\mathbf{a}\|\|\mathbf{b}\|$, so the inequality also holds in this case.

5. *Consider the following system of linear equations.*

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ x & + & 2y & + & z & = & 3 \\ 2x & + & 3y & + & 2z & = & 5 \end{array}$$

Interpret the planes and their intersection lines. Describe all the solutions of the system.

Megoldás: The first two planes are not parallel because the normal vectors, $(1,1,1)$ and $(1,2,1)$ are not parallel. This means that their intersection is a line. However, since the third equation is the sum of the previous two equations, every point of the intersection line of the first two is also in the third plane. On the other hand, the third is not parallel to any of the others, so these are three different planes having a common line.

Actually, it is even more apparent that these planes cannot intersect in a single point if we consider the system of equations columnwise that is, the question would be to find the possible linear combinations of the vectors $\mathbf{v}_1 = (1,1,2)$, $\mathbf{v}_2 = (1,2,3)$ and $\mathbf{v}_3 = (1,1,2)$ which give $(2,3,5)$. Since two of the three vectors are equal, they only span a plane, so $(2,3,5)$ may not even be in it. But if is, then when we write it az a linear combination of $(1,1,2)$ and $(1,2,3)$ then we can separate the first coefficient in infinitely many ways to get the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_3$.

To solve the system, we use the elimination method first. The pivot element will be in boldface.

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \end{array}\right] \mapsto \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the modified system is

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ & & y & & & = & 1 \\ & & & & 0 & = & 0. \end{array}$$

Thus $y = 1$, z can be chosen arbitrarily (say, $z = t$ for some $t \in \mathbb{R}$), and then by substitution we get that $x = 1 - t$. These satisfy all three of the equations, so the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad (t \in \mathbb{R}),$$

a line which contains the point $(1,1,0)$ (when $t = 0$), and is parallel with the vector $(-1,0,1)$.

6. *Consider the following two systems of linear equations (c a real parameter).*

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 1 \\ x & - & 2y & - & z & = & -2 \\ 2x & - & 3y & + & cz & = & 1 \end{array} \qquad\qquad \begin{array}{rcrcrcr} x & - & y & + & z & = & 1 \\ 2x & - & 2y & - & z & = & -1 \\ 2x & + & cy & + & z & = & 3 \end{array}$$

How does the number of solutions depend on c in each system?
(The second system is not quite the same as in the original problem sheet. This way you can see more different cases.)

Solution: If we bring the matrix of the system of equations into row echelon form, we can tell from its shape the number of solutions without actually calculating the solutions.

$$\left[\begin{array}{ccc|c} \mathbf{1} & -1 & 1 & 1 \\ 1 & -2 & -1 & -2 \\ 2 & -3 & c & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & \mathbf{-1} & -2 & -3 \\ 0 & -1 & c-2 & -1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & c & 2 \end{array}\right]$$

This is a full triangle if $c \neq 0$, so in that case there is one solution. However, if $c = 0$ then the last equation is contradictory, so in that case there is no solution.

$$\left[\begin{array}{ccc|c} \mathbf{1} & -1 & 1 & 1 \\ 2 & -2 & -1 & -1 \\ 2 & c & 1 & 3 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & c+2 & -1 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & c+2 & -1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

If $c \neq -2$ then we got a full triangle in the coefficient matrix, so there is one solution. If $c = -2$ then the second and the third equation contradicts to each other. It is more apparent if we continue with the elimination in the third column.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & \mathbf{-1} & 1 \\ 0 & 0 & -3 & -3 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

There is a contradiction in the third row, so there is no solution in this case.

8. *Construct a 3×3 system that needs two row exchanges to reach a triangular form.*

Solution: The system of equations whose augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

is such a system. Here for the first pivot we have to switch the first and the third equation (and we don't need here any elimination step), but then there is still no pivot in the second row, so we need another row exchange.

9. *Suppose the first two columns are the same. What happens during elimination?*

Solution: After eliminating the nonzero elements under the first pivot, the elements of the second column starting from the second row become all 0's, so we cannot find a pivot in the second column. This matrix will have no full triangular form but it can still have a row echelon form where the first nonzero element (that is, the leading element) of each row is to the right of the leading element of the previous row, and we can also describe the (infinitely many, if there is a solution at all) solutions of the system, as in problem 5.

10. *For which numbers a does the elimination stop before reaching a full triangle in*

$$\begin{bmatrix} a & 1 \\ a & a \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}?$$

Solution: If $a = 0$ then then there is no pivot in the first column. If $a \neq 0$ then after the first elimination in the 2×2 matrix we get $\begin{bmatrix} a & 1 \\ 0 & a-1 \end{bmatrix}$, which is a full triangle if $a \neq 1$. But for $a = 1$ we get $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, which is again not a full triangle. So the 2×2 matrix has a full triangle form if and only if $a \neq 0$ and $a \neq 1$.
For the second, we get

$$\begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \mapsto \begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{bmatrix} \mapsto \begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{bmatrix},$$

which is a full triangle if $a \neq 0, 2, 4$, but the elimination process stops in the first column if $a = 0$, and in the second if $a = 2$.

6'. *To give a fuller picture of the possibilities, here is another variant of the questions in problem 6. The question is again the number of solutions, depending on the parameter c.*

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 1 \\ 2x & - & 2y & - & z & = & -1 \\ 2x & + & cy & + & z & = & 1 \end{array}$$

Solution:

$$\left[\begin{array}{rrrcr} \mathbf{1} & -1 & 1 & | & 1 \\ 2 & -2 & -1 & | & -1 \\ 2 & c & 1 & | & 1 \end{array}\right] \mapsto \left[\begin{array}{lrrcr} 1 & -1 & 1 & | & 1 \\ 0 & 0 & -3 & | & -3 \\ 0 & c+2 & -1 & | & -1 \end{array}\right] \mapsto \left[\begin{array}{lrrcr} 1 & -1 & 1 & | & 1 \\ 0 & c+2 & -1 & | & -1 \\ 0 & 0 & -3 & | & -3 \end{array}\right]$$

If $c \neq -2$ then we got a full triangle in the coefficient matrix, so there is one solution. If $c = -2$ then there is no possible pivot element in the second column. But we can continue with the elimination, using the third column.

$$\left[\begin{array}{lrrcr} 1 & -1 & 1 & | & 1 \\ 0 & 0 & -\mathbf{1} & | & -1 \\ 0 & 0 & -3 & | & -3 \end{array}\right] \mapsto \left[\begin{array}{lrrcr} 1 & -1 & 1 & | & 1 \\ 0 & 0 & -1 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{array}\right]$$

Here there is no row that would be contradictory in itself. On the other hand, if we give an arbitrary value to y (the only unknown whose column contains no pivot element) then expresing z from the second row, and after substitution, x from the first, we get infinitely many different solutions. (They satisfy the last equation because that poses no restrictions on the solutions, and also the first and second because we calculated z and x so that they would be satisfied.)