1. *Let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. *Write down the three elimination matrices* E_{21}, E_{31} *and* E_{32} *that turn* A *into an upper triangular form. Which of these three matrices commute with each other? Compute the matrix product* $E_{32}E_{31}E_{21}A$.

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The row operations were: subtract 2 times the first row from the second, subtract 1 times the first row from the third, and subtract 1 times the second row from the third, so the elimination matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The products of the elimination matrices are:

$$E_{21}E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$$E_{21}E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \neq E_{32}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix},$$

$$E_{31}E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = E_{32}E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix},$$

so E_{21} and E_{32} do not commute, the other pairs do. This can also be seen from the row operations: at the first and the second, and also at the second and the third, the row whose multiple we subtract is not changed by any of the two operations, but if we apply E_{32} after E_{21} there we subtract the already changed second row, so it is different from applying E_{32} first, and then E_{21}.

$$E_{32}E_{31}E_{21}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = E_{32}E_{31}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} = E_{32}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

2. *True or false?*

(i) If every entry of A *and* B *is positive then every entry of* AB *is also positive. (We assume* AB *exists.)*

(ii) If elimination takes A *to* U *then* $A\mathbf{x} = \mathbf{0}$ *implies* $U\mathbf{x} = \mathbf{0}$.

(iii) If $A\mathbf{x} = \mathbf{0}$ *has a nontrivial solution then* A *has no* 0 *entries.*

(iv) If AB *and* BA *are both defined then both of* AB *and* BA *are square.*

Solution: (i) True: the j'th element of the i'th row is $\sum_t a_{it}b_{tj} > 0$, because every summand is positive.

(ii) True: the same elimination steps take the augmented matrix $[A|\mathbf{0}]$ of the system of equations $A\mathbf{x} = \mathbf{0}$ to $[U|\mathbf{0}]$, so the solutions of $U\mathbf{x} = \mathbf{0}$ are the same as the solutions of $A\mathbf{x} = \mathbf{0}$.

(iii) False: for example $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(iv) True: Let A be $m \times n$ and B be $k \times \ell$. Then AB is defined if $n = k$ and BA is defined if $\ell = m$. So B is an $n \times m$ matrix, hence AB is $m \times m$, and BA is $n \times n$.

3. *Determine the "swap matrix" S_{ij} such that $S_{ij}A$ is the same as A, but its rows i and j are swapped. What is AS_{ij} (if A is $n \times n$)?*

Solution: For $k \neq i, j$, the k'th row of S_{ij} has only one 1, and it is in the diagonal of the matrix. The i'th row only has a 1 at the j'th position, since the multiplication by S_{ij} writes the j'th row of A in its i'th row, and similarly, the j'th row has only one 1, at the i'th position. All the other elements of the rows are 0.
Multiplication by S_{ij} from the right makes a column operation, which leaves every column but the i'th and j'th unchanged, and swaps the i'th and j'th column because in the i'th column of S_{ij} there is a 1 only at the j'th row, and in the j'th column only at the i'th row.

4. *Multiply the following matrices A and B. First using row-times-column dot products and then using column-times-row matrices that you add up.*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 1 \cdot 1 + 1 \cdot 0 & 1\cdot(-1) + 1\cdot(-1) + 1\cdot(-2) \\ 2\cdot 1 + 1 \cdot 1 + 1 \cdot 0 & 2\cdot(-1) + 1\cdot(-1) + 1\cdot(-2) \\ 1\cdot 1 + 0 \cdot 1 + 0 \cdot 0 & 1\cdot(-1) + 0\cdot(-1) + 0\cdot(-2) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 3 & -5 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 3 & -5 \\ 1 & -1 \end{bmatrix}$$

5. *Let E_{21} denote the elimination matrix that subtracts the double of row 1 from row 2 and let S_{23} denote the swap matrix of rows 2 and 3. What is $S_{23}E_{21}$, which does the two steps at once?*

Solution:

$$S_{23}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

The resulting operation is: moving the third row to the position of the second, and writing the second row minus twice the first row in the third row.

6. *Write the following problem as a system of linear equations $A\mathbf{x} = \mathbf{b}$: Old Smith has a son who is two years younger than half of his age. He also has a granddaughter, who is 4 years older than the third of her father's age. The sum of all of their ages is 118. How old are they?*

Solution: Let x be the age of Old Smith, y the age of his son, and z the age of his granddaughter. Then we get the equations: $y = \frac{1}{2}x - 2$, $z = \frac{1}{3}y + 4$ and $x + y + z = 118$. So the system of equations we get is

$$\begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 0 & \frac{1}{3} & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 118 \end{bmatrix}.$$

The elimination gives

$$\left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 2 \\ 0 & \frac{1}{3} & -1 & -4 \\ 1 & 1 & 1 & 118 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 2 \\ 0 & \frac{1}{3} & -1 & -4 \\ 0 & 3 & 1 & 114 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} \frac{1}{2} & -1 & 0 & 2 \\ 0 & \frac{1}{3} & -1 & -4 \\ 0 & 0 & 10 & 150 \end{array}\right],$$

so $z = 15$, $y = 33$, $x = 70$.

7. *Multiply the following matrices C and D as CD and as DC. Also compute C^{10} and D^{200}.*

$$C = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

Solution:

$$CD = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, \qquad DC = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$$

Since the row operation belonging to C does not change the first row, whose multiples we add to the other two rows, if we apply it 10 times, we get the same as when we add $10a$ times the first row to the second, and $10b$ times the first row to the third. Similarly, the elimination belonging to D does not change the second row, so applying D 100 times is the same as adding $100c$ times the second row to the third. The corresponding elimination matrices are

$$C^{10} = \begin{bmatrix} 1 & 0 & 0 \\ 10a & 1 & 0 \\ 10b & 0 & 1 \end{bmatrix}, \qquad D^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 100c & 1 \end{bmatrix}$$

8. *Show that for $n \times n$ matrices A, B in general we do not have $(A+B)^2 = A^2 + 2AB + B^2$. What should we write instead of the middle term?*

Solution: $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$, and this is in general not the same as $A^2 + 2AB + B^2$, because BA is usually not equal to AB. For example, for $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ we have $AB = \begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$. And

$$(A+B)^2 = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & 8 \\ -4 & 2 \end{bmatrix},$$

while

$$A^2 + 2AB + B^2 = \begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ -3 & 2 \end{bmatrix}$$

9. **HW** *(3×3 matrices) What is the matrix B such that for every matrix A*
(i) $BA = 3A$;
(ii) every row of BA is equal to the first row of A?

10. *For which numbers a, b, c, d does the following hold?*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix},$$

so their equality gives the system of equations $a+b = a+c$, $a+b = b+d$, $c+d = a+c$, $c+d = b+d$, that is, $b = c$, $a = d$, $d = a$ and $c = b$. This means that only the matrices $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ satisfy this condition, where $a, b \in \mathbb{R}$ is arbitrary.