1. *Compute the inverse of* $\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ *using Gauss–Jordan's method.*

Solution:

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 2 & 0 & 4 & -2 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array}\right],$$

so the inverse matrix is $\begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$.

2. *True or false?*

(i) If $A^T = A^{-1}$ then A is symmetric.
(ii) If $A^T A = AA^T$ then A is square.
(iii) If $A^T = -A$ then A is invertible.
(iv) Every elimination matrix is diagonally dominant, so invertible.
(v) Suppose in the 3×3 matrix A the sum of the first two columns is the third column. Then $A\mathbf{x} = \mathbf{0}$ surely has a nonzero solution.

Solution:

(i) False. $A^T = A^{-1}$ holds, for example, for $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ but the matrix is not symmetric.
(ii) True. If A is $m \times n$ then $A^T A$ is $n \times n$, while AA^T is $m \times m$, so if they are equal then $m = n$.
(iii) False. $A^T = -A$ is even true for the zero matrix.
(iv) False. Though they are invertible but not necessarily diagonally dominant, for example, $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ is a not diagonally dominant elimination matrix.
(v) True. $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a nontrivial solution.

3. *Determine the inverse of the following two matrices* $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$.

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I|A^{-1}] \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[B|I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \mapsto$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right] = [I|B^{-1}]$$

$$\Rightarrow \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

4. *Suppose A, B, C are $n \times n$ and $ABC = D$ is invertible. Show that A, B, C are all invertible and express B^{-1} using D^{-1} and A, C.*

Solution: $D^{-1}ABC = D^{-1}D = I$, so C is invertible, and $C^{-1} = D^{-1}AB$. But then $I = CC^{-1} = CD^{-1}AB$, so B is invertible, and $B^{-1} = CD^{-1}A$. Finally, A is invertible, since $I = BB^{-1} = BCD^{-1}A$.

5. *What is a, b so that the following holds?*

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

Can you generalise?

Solution:

$$I = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix},$$

and the elements of the product matrix are $4a - 3b$ in the diagonal, and $2b - a$ everywhere else. So $4a - 3b = 1$ and $2b - a = 0$, which gives $a = \frac{2}{5}$ and $b = \frac{1}{5}$. So the inverse of the matrix which has 4 in the diagonal and -1 elsewhere is the matrix which has $\frac{2}{5}$ in the diagonal and $\frac{1}{5}$ elsewhere.

If our original matrix were $n \times n$ with n in the diagonal and -1 everywhere else, then the inverse will also be a matrix which has a's in the diagonal and b's elsewhere, and a, b satisfy the equations $na - (n-1)b = 1$ and $-a + 2b = 0$, which gives $a = \frac{2}{n+1}$ and $b = \frac{1}{n+1}$.

The reason behind this is that an $n \times n$ matrix A which is uniform in the diagonal and also outside the diagonal can be written as a linear combination of the matrix I and J, where J is the matrix whose every entry is 1. Then $J^2 = nJ$ and the inverse of A will also be a linear combination of I and J, and it can be found by solving the linear equation $(aI + bJ)(xI + yJ) = axI + (bx + ay + nby)J = I$, that is, $x = a^{-1}$, and $y = -ba^{-1}/(a + nb)$.

6. *Compute the LU decomposition of*

$$C = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Make it into an LDU decomposition.

Solution:

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 3 & 1 & 0 \\ 0 & 8/3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 3 & 1 & 0 \\ 0 & 8/3 & 1 \\ 0 & 0 & 21/8 \end{bmatrix} = U, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & 3/8 & 1 \end{bmatrix}$$

So the LU decomposition is

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & 3/8 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 8/3 & 1 \\ 0 & 0 & 21/8 \end{bmatrix},$$

and from this the LDU decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & 3/8 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8/3 & 0 \\ 0 & 0 & 21/8 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 3/8 \\ 0 & 0 & 1 \end{bmatrix}.$$

7. *Separate the following system* $A\mathbf{x} = \mathbf{b}$ *into* $L\mathbf{y} = \mathbf{b}$ *and* $U\mathbf{x} = \mathbf{y}$.

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ x & + & 2y & + & 3z & = & 3 \\ x & + & 3y & + & 5z & = & 10. \end{array}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = U, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

So the two systems of equations are

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{y}.$$

The first has a solution $\mathbf{y} = (1, 2, 5)$ but the second is contradictory. (The original system has no solution, either.)

8. HW *Let* $A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$. *For which values of* c *does elimination break down and how? Write* $A = LU$ *in the other cases.*

9. *We mentioned that if* A, B *are lower triangular then their product* AB *is also lower triangular. Prove it. Also prove that if* A *is lower triangular with nonzero elements on the diagonal then the inverse* A^{-1} *is also lower triangular.*

Hint for both: Consider the i*-th column of* AB *being governed by the* i*-th column of* B*. Where are the* 0*'s?*

Solution: The i-th column of AB is the linear combination of the columns of A with coefficients taken from the i-th column of B. But the latter can have nonzero elements only in the entries on and below the diagonal, so the result is a linear combination of columns of A starting at the i-th column. However, these columns have only 0's before the i'th entry, so this will be true for their linear combination, as well.

For the inverse, we are looking for a matrix B such that $AB = I$. We get the i-th column of I as a linear combination of columns of A. Suppose that the k-th is the first column which actually appears in this linear combination. Then the k-th entry of the product will be nonzero, since the remaining columns are all 0 at this entry. But this means that $k \geq i$, so we get the i-th column of AB as the linear combination of the columns of A starting with the i-th, that is, the first $i - 1$ entries of the i-th column of B are 0. This proves that $B = A^{-1}$ must be a lower triangular matrix.

Another proof for the statement about the inverse is that such a matrix can be changed to a diagonal matrix by using only downward elimination (that is, left multiplication by lower triangular matrices) and then to I by scalar multiplication of the rows (this means left multiplication by diagonal matrices). So $I = X_1 \cdots X_m A$ for some lower triangular matrices, thus $A^{-1} = X_1 \cdots X_m$ is also lower triangular by the first part of the problem.