1. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

a) Is there a subspace of the vector space of 2×2 matrices that contains exactly one of A and B?
b) Is there a subspace that contains exactly two of A, B and I?
c) Is there a subspace that contains no nonzero diagonal matrices?

Solution: a) Yes, there is, actually in both ways:
$\text{span}(A) = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\}$ does not contain B, and $\text{span}(B) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & -y \end{bmatrix} \,\middle|\, y \in \mathbb{R} \right\}$ does not contain A.

b) No, there is no such subspace, since $A - B = I$, so $I \in \text{span}(A, B)$, and from this equation we also get $A = B + I$ and $B = A - I$, thus $A \in \text{span}(B, I)$ and $B \in \text{span}(A, I)$.

c) Yes, there is, for example $\text{span}(C)$, where $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

2. *True or false for an $m \times n$ matrix A of rank r describing a system $A\mathbf{x} = \mathbf{b}$?*
(i) If $n = m = r$ then A is invertible.
(ii) If $n < m$ then the system has no solution.
(iii) If $n > m$ then there are free columns.
(iv) If the system has a unique solution then $n = r$.
(v) If the system has no solution then $\mathbf{b} \neq \mathbf{0}$ and $r > 0$.

Solution: (i) True. If $n = m = r$ then the reduced row echelon form contains $m = n$ nonzero rows, so every row and every column has a pivot, thus the reduced row echelon form is I.
(ii) False, for example the system $x = 2$, $2x = 4$ has a solution, and here $m = 2$, $n = 1$.
(iii) True, since there can be at most m pivots, so there must be a column is the row echelon form which contains no pivot.
(iv) True. There cannot be a free column, so every column contains a pivot in the row echelon form. On the other hand, there cannot be more pivots than n, so the number of pivots (and then also the number of nonzero rows in a row echelon form, that is, the rank of A) is n.
(v) False. Though $\mathbf{b} \neq \mathbf{0}$ does follow but r may be zero: $0x = 1$ has no solution.

3. *Let $V = \mathbb{R}^2$ but scaling is defined by $\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ 0 \end{bmatrix}$. Is this a vector space?*

Solution: No, the axiom $1\mathbf{v} = \mathbf{v}$ does not hold. (Actually, this is the only axiom that fails here.)

4. *Let $V = \mathbb{R}^2$ but addition is defined by $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_2 \\ v_2 + w_1 \end{bmatrix}$. Is this a vector space?*

Solution: No, here the addition is not commutative:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_2 \\ v_2 + w_1 \end{bmatrix} \quad \text{but} \quad \mathbf{w} + \mathbf{v} = \begin{bmatrix} w_1 + v_2 \\ w_2 + v_1 \end{bmatrix} = \begin{bmatrix} v_2 + w_1 \\ v_1 + w_2 \end{bmatrix},$$

so $\mathbf{v} + \mathbf{w} \neq \mathbf{w} + \mathbf{v}$ even for $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
(The associativity of the addition also fails).

5. *Describe the smallest subspace of the 2×3 matrices that contain*

$$(a)\ A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad (b)\ A \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}; \quad (c)\ B \text{ and } C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution:
a) $\{\, xA \mid x \in \mathbb{R} \,\} = \left\{ \begin{bmatrix} x & x & x \\ 0 & 0 & 0 \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\}$, that is, the set of matrices whose second row is all 0, and the entries of the first row are equal.

b) $\{\, xA + yB \mid x, y \in \mathbb{R} \,\} = \left\{ \begin{bmatrix} x + y & x + 2y & x + 3y \\ 0 & 0 & 0 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$. These are the matrices whose second row is all zero, and the first row forms an arithmetic sequence (it is clearly true for these matrices, and for any sequence $a, a + d, a + 2d$, we have $y = d$ and $x = a - d$).

c) $\{ xB + yC \,|\, x, y \in \mathbb{R} \} = \left\{ \begin{bmatrix} x+y & 2x+2y & 3x+3y \\ y & 2y & 3y \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$. These are the matrices whose every row is a scalar multiple of $(1,2,3)$ (indeed, if the rows are c times and d times $(1,2,3)$ then $y = d$ and $x = c - d$ works).

6. *Which of the following give the correct definition of the rank of A (with R being a reduced row echelon form)?*

(i) The number of nonzero rows of R.
(ii) The number of columns minus the number of zero rows.
(iii) The number of columns minus the number of free columns.
(iv) The number of 1's in the matrix R.

Solution: (i) is the definition of rank. But (iii) is also equal to the rank: the difference of the number of columns and the number of free columns is the number of pivot columns, that is, the number of pivots in R, and this is equal to the number of nonzero rows in R.

(ii) is not equal to the rank, for example for $A = R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(iv) is not equal to the rank, either. The number of pivots in R is equal to the rank but there can be other 1's in R, the matrix A above is also a counterexample here.

7. *Write the special solutions of $R\mathbf{x} = \mathbf{0}$ and of $R^T\mathbf{y} = \mathbf{0}$ for the following matrices. Write down the nullspace matrices.*

$$R_1 = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: R_1 is in reduced row echelon form, where the first and second are the pivot columns. We have special solutions assigned to each free column by giving the value 1 to the corresponding free variable and 0 to the other free variable(s), then expressing the pivot variables. We get the special solution corresponding to the free variable x_i by writing 1 in the i'th position, 0 in the other "free" positions and then filling out the gaps by the negatives of the first r entries of the i'th column of R. So the special solution corresponding to x_3 is $(-2,-3,1,0)$, the one corresponding to x_4 is $(-4,-5,0,1)$. (Check that these are indeed solutions of $R_1\mathbf{x} = \mathbf{0}$.) The nullspace matrix contains these special solutions as columns:

$$\begin{bmatrix} -2 & -4 \\ -3 & -5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

R_2 is also reduced row echelon, and its only pivot column is the second. The special solution corresponding to x_1 is $(1,0,0)$, to x_3 is $(0,-2,1)$. The nullspace matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$. R_1^T and R_2^T are not reduced row echelon matrices, so we have to use elementary row operations to bring them to reduced row echelon form.

$$R_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 5 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The free variable is y_3, the special solution is $(0,0,1)$, the nullspace matrix is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$R_2^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The free variables are y_2, y_3, the special solutions are $(0,1,0)$ and $(0,0,1)$, and the nullspace matrix is $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

8. **HW** *Find the reduced row echelon form and the rank of* $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 1 & c & 2 \end{bmatrix}$. *Which are the pivot columns? Give the special solutions. (The answer will depend on c.)*

9. *Prove that every rank-r matrix can be written as a sum of* r *rank-*1 *matrices.*

Solution: If we bring A to row echelon form, we get $PA = R$, where P is the product of the row operation matrices, so it is invertible. Then $A = P^{-1}R$. If calculate this product as the sum of column times row matrices, then only the first r summands will be nonzero, since the other rows of R are zero. So we decomposed A into a sum of r rank-1 matrices.
(In fact, we can also write the $m \times n$ matrix A as the product of the $m \times r$ matrix consisting of the pivot columns of A and the $r \times n$ matrix consisting of the nonzero rows of the reduced row echelon form, and this obviously gives a sum decomposition into r rank-1 matrices if we use the column times row products. This is a more convenient way to decompose A.)