1. *Let* $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$. *Show that the first three columns are independent, but all four are dependent. Describe the dependencies using the equation* $A\mathbf{v} = \mathbf{0}$. *Show that the four columns of* A *span* $\mathbb{R}^3$. *Do they form a basis? Which columns form a basis?*

Solution: Since this is a row echelon matrix, with pivots in the first three columns, we can see immediately that the first three columns are independent, and the fourth depends on them. To see which linear combination of the first three columns gives the fourth, we calculate the reduced row echelon form.

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Considering A as the augmented matrix for the equation $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 = \mathbf{c}_4$ (where $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ and $\mathbf{c}_4$ are the columns of A), we can see that $\mathbf{c}_4 = -\mathbf{c}_1 - \mathbf{c}_2 + 4\mathbf{c}_3$.
Since $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$ are three independent vectors in the 3-dimensional space $\mathbb{R}^3$, they span the whole space, so the four columns also span $\mathbb{R}^3$. But the four columns do not form a basis, since they are not independent.
In a 3-dimensional space, like $\mathbb{R}^3$, every basis has 3 elements, and 3 elements form a basis if and only if they span the whole space. The equation $-\mathbf{c}_1 - \mathbf{c}_2 + 4\mathbf{c}_3 - \mathbf{c}_4 = \mathbf{0}$ also shows that each vector can be expressed as a linear combination of the other three, that is, any of them can be omitted from the four-element spanning set. So any three form a spanning set, hence also a basis.

2. *True or false?*
(i) A *and* A^T *have the same number of pivots.*
(ii) A *and* A^T *have the same left nullspace.*
(iii) If the row space of A *equals the columns space of* A *then* $A = A^T$
(iv) If $A = -A^T$ *then the row space of* A *equals the column space of* A
(v) The columns of A *form a basis of* $C(A)$.

Solution: (i) True, because the rank of A is the same as the rank of A^T.
(ii) False. They may not even be in the same vector space.
(iii) False. For an invertible $n \times n$ matrix both the row space and the column space are $\mathbb{R}^n$ but the matrix may not be symmetric (say, $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$).
(iv) True, since the row space of A is the same as the column space of A^T, and the columns of A span the same subspace as their negatives, that is, the columns of $-A^T = A$.
(v) False. The columns may be dependent: $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is a counterexample.

3. *Which of the columns form a basis of* $C(R)$*? Describe the basis of* $N(R)$. *Answer these questions for* R^T, *too.*

$$R = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 3 & -5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: Since R is a row echelon matrix, we can immediately tell which are the pivot columns: the first, second and fourth, so they form a basis $\{(2, 0, 0, 0), (0, 2, 0, 0), (4, -5, -1, 0)\}$ of $C(R)$. But to find the basis of $N(R)$ we bring the matrix to reduced row echelon form, and find the special solutions of $R\mathbf{x} = \mathbf{0}$.

$$R = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 3 & -5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There is one free column, the third, and the special solution corresponding to x_3 is $(-1, -\frac{3}{2}, 1, 0)$, so this vector alone forms a basis of $N(R)$. (Or we can choose a scalar multiple instead: $(2, 3, -2, 0)$.)

$$R^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & -5 & -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -5 & -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the first three columns form a basis $\{(2, 0, 2, 4), (0, 2, 3, -5), (0, 0, 0, -1)\}$ of the column space of R^T. The special solution of $R^T\mathbf{x} = \mathbf{0}$ is $(0, 0, 0, 1)$, so $\{(0, 0, 0, 1)\}$ is a basis of $N(R^T)$.
(Actually, if R is an $m \times n$ row echelon matrix of rank r then the first r rows of R always form a basis of the row space of R, so also of the column space of R^T, and the standard basis vectors $\mathbf{e}_i$ $(i = r+1, \dots, m)$ form a basis of the left null space of R, that is, the null space of R^T, since they are clearly independent, orthogonal to the rows of R, and their number is the same as $\dim N(R^T) = m - r(R^T) = m - r(R) = m - r$. However, if R is row equivalent to A then $N(R^T)$ and $N(A^T)$ are not the same, so in general, when we want to determine the basis of the four fundamental spaces, we cannot avoid doing the elimination for both A and A^T.)

4. *Find the complete solutions of the following systems of equations.*

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 3 & -1 & 1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -1 & 1 & -2 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 3 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The pivot variables are x_1, x_3, and the only free variable is x_2. A particular solution is $\mathbf{x}_p = (5, 0, -3)$, the special solution for the null space is $(1, 1, 0)$, so

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 & 5 \\ 1 & 3 & -2 & -1 & 4 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 3 & -3 & -3 & 3 \\ 0 & 3 & -3 & -3 & 3 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The pivot variables are x_1, x_2, the free variables x_3, x_4. The particular solution is $(1, 1, 0, 0)$, the special solutions of the null space corresponding to x_3 and x_4 are $(-1, 1, 1, 0)$ and $(-2, 1, 0, 1)$. So

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

5. *Let $M = M^{3\times 3}$, the vector space of real 3×3 matrices.*
(i) Let V denote the subspace of M spanned by the diagonal matrices. What is V? What is $\dim V$*?*
(ii) Let W denote the subspace of M spanned by the rank-1 matrices. What is W? What is $\dim W$*?*

Solution: (i) A scalar multiple or sum of diagonal matrices is also diagonal, so the spanned subspace is actually the set of diagonal matrices. Its dimension is 3, since every diagonal matrix can be written uniquely as a linear combination of the matrices which have only one nonzero element: a 1 somewhere in the diagonal.

(ii) Every element A of M can be written as a sum of rank-1 matrices, for example the matrices that keep only one row of A, and replace the other rows with $\mathbf{0}$ rows. So the spanned subspace is the whole M, which is 9 dimensional.

6. *Construct many independent 3×3 matrices of row echelon form. How many can you find?*

Solution: A row echelon matrix is always upper triangular (it is not true the other way around), so in a 3×3 row echelon matrix A, the elements a_{21}, a_{31} and a_{32} are all zero. This means that such matrices are always in the subspace V of upper triangular matrices of $\mathbb{R}^{3\times 3}$, and in V the six upper triangular matrices that have only one 1 entry, and the others are 0, form a basis, so this subspace cannot have more than six independent elements (note that most of these basis elements of V are not of row echelon form). On the other hand, we can find six independent row echelon matrices:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These matrices are independent, since $\sum_{i=1}^{6} x_i A_i = 0$ implies that $x_1 + x_4 + x_5 + x_6 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 + x_6 = 0$, $x_5 = 0$ and $x_6 = 0$, and then $x_i = 0$ for $i = 1, \ldots, 6$. So the maximum number of independent row echelon matrices is 6 (and also for the reduced row echelon matrices, since $A_1, \ldots, A_6$ are actually reduced row echelon matrices).

7. *You are allowed to put four 1's into a 3×3 matrix, the rest of the entries are 0. How to do this if you want to keep the dimension of the column space as small as possible? How to do this if you want to keep the dimension of the row space as small as possible? How to do this if you want to keep the dimension of the nullspace as small as possible? What is the minimum of the sum of the dimensions of all the Four Fundamental Subspaces?*

Solution: Since the matrix is not the zero matrix, its rank (which is equal to the dimension of the row space and also to the dimension of the column space) is at least one, and we can easily find a rank-1 matrix containing four 1's, and the others 0, as the matrix A below.

The null space can even be 0 dimensional if we make the matrix invertible, the full triangular matrix B is such an example.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, the sum of dimensions of the column space and null space of any 3×3 matrix (and then also of its transposed matrix) is 3, so the sum of the dimensions of the four fundamental subspaces is always 6.

8. *Construct a matrix with the following property or refute the possibility.*

(i) $C(A)$ *contains* $(1,1,0)$, $(0,0,1)$ *and* $C(A^T)$ *contains* $(1,2)$ *and* $(2,5)$.
(ii) $C(A)$ *has basis* $\{(1,1,3)\}$ *and* $N(A)$ *has basis* $\{(3,1,1)\}$.
(iii) **HW** $\dim N(A) = 1 + \dim N(A^T)$.
(iv) **HW** $N(A)$ *contains* $(1,2)$, $C(A)$ *contains* $(2,1)$.
(v) **HW** $C(A) = C(A^T)$ *but* $N(A) \neq N(A^T)$.

Solution: (i) It follows from the conditions that $C(A) \leq \mathbb{R}^3$ and the row space $C(A^T) \leq \mathbb{R}^2$, so A is a 3×2 matrix. On the other hand, $(1,2)$ and $(2,5)$ form a basis in $\mathbb{R}^2$, so the rank of A must be 2, moreover, any 3×2 matrix with rank 2 satisfies the second condition. Thus $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies both.

(ii) The conditions imply that $C(A) \leq \mathbb{R}^3$ and $N(A) \leq \mathbb{R}^3$, so A must be a 3×3 matrix. But then $\dim C(A) + \dim N(A) = r(A) + \dim N(A) = 3$. However, according to the conditions, $\dim C(A) = 1$ and $\dim N(A) = 1$, so there is no such matrix.

9. *Verify that $C(AB) \leq C(A)$ and conclude that $r(AB) \leq r(A)$. Show similarly that $r(AB) \leq r(B)$.*

Solution: Since the columns of AB are linear combinations of the columns of A, we have $C(AB) \leq C(A)$, so $r(AB) = \dim C(AB) \leq \dim C(A) = r(A)$. Similarly, the rows of AB are linear combinations of the rows of B, so the row space of AB is a subspace of the row space of B, which gives the inequality between their dimensions: $r(AB) \leq r(B)$.