1. *Let* $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \end{bmatrix}$. *Determine bases for each of* $C(A)$, $C(A^T)$, $N(A)$, $N(A^T)$. *Check the orthogonality of the respective subspaces.*

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the reduced row echelon form of A we can see that
$\{(1,0,1),(1,1,2)\}$ (the pivot columns of A) is a basis for $C(A)$,
$\{(1,0,0,-1),(0,1,1,3)\}$ (the nonzero rows of $rref(A)$ is a basis for $C(A^T)$, and
$\{(0,-1,1,0),(1,-3,0,1)\}$ (the special solutions of $A\mathbf{x} = \mathbf{0}$) is a basis for $N(A)$.
For $N(A^T)$ we use $rref(A^T)$.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\{-1,-1,1)\}$ (the special solution of $A^T\mathbf{x} = \mathbf{0}$) is a basis for $N(A^T)$.
To see that two subspaces are orthogonal, it is enough to check that their spanning elements are orthogonal.
$C(A) \perp N(A^T)$: $(1,0,1)(-1,-1,1) = -1+1 = 0$, $(1,1,2)(-1,-1,1) = -1-1+2 = 0$.
$C(A^T) \perp N(A)$: $(1,0,0,-1)(0,-1,1,0) = 0+0+0+0 = 0$, $(1,0,0,-1)(1,-3,0,1) = 1+0+0-1 = 0$, $(0,1,1,3)(0,-1,1,0) = 0-1+1+0 = 0$, $(0,1,1,3)((1,-3,0,1) = 0-3+0+3 = 0$.

2. *True or false?*
(i) A *and* A^T *are orthogonal to each other.*
(ii) $N(A)$ *and* $N(A^T)$ *are orthogonal complements.*
(iii) If the row space of A *equals the column space of* A *then* $N(A) = N(A^T)$.
(iv) The projection matrix on $N(A^T)$ *is* $I - A(A^TA)^{-1}A^T$ *if* A *has independent columns.*
(v) If $C(A)$ *and* $N(A)$ *are orthogonal complements then* $A = A^T$.

Solution: (i) False. We defined orthogonality for subspaces not for matrices. But even if we consider $A \in \mathbb{R}^{m\times n}$ and $A^T \in \mathbb{R}^{n\times m}$ as vectors of $\mathbb{R}^{mn}$, we almost never get orthogonal vectors, for instance, for an all positive A, these vectors cannot be orthogonal.
(ii) False. For an $m \times n$ matrix $N(A) \leq \mathbb{R}^n$ and $N(A^T) \leq \mathbb{R}^m$, so these subspaces are usually not even in the same vector space.
(iii) True. If $C(A^T) = C(A)$ then $N(A) = C(A^T)^\perp = C(A)^\perp = N(A^T)$.
(iv) True. $A(A^TA)^{-1}A^T$ is the projection matrix on $C(A)$, so $I - A(A^TA)^{-1}A^T$ is the projection matrix on its orthogonal complement, $N(A^T)$.
(v) False. For example, if A is an invertible $n\times n$ matrix, which is not symmetric, say, $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then $C(A) = \mathbb{R}^n$ and $N(A) = \{\mathbf{0}\}$ orthogonal complements but $A \neq A^T$.

3. *The following system of equations has no solution:*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 1 & 3 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} \qquad A\mathbf{x} = \mathbf{b}.$$

Find a combination of the equations that produces $0 = 1$. *Reinterpret this to finding a vector* $\mathbf{y} \in N(A^T)$ *such that* $\mathbf{y}^T\mathbf{b} = 1$.
Prove that given an $m \times n$ *matrix* A *and a vector* $\mathbf{b} \in \mathbb{R}^m$, *either there exists* $\mathbf{x} \in \mathbb{R}^n$ *such that* $A\mathbf{x} = \mathbf{b}$, *or there exists* $\mathbf{y} \in N(A^T)$ *such that* $\mathbf{y}^T\mathbf{b} = 1$.

Solution: The Gaussian elimination

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 2 & 3 & -1 & | & 4 \\ 1 & 3 & -2 & | & 4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 3 & -3 & | & 2 \\ 0 & 3 & -3 & | & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 3 & -3 & | & 2 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

shows that $(\mathbf{r}_3-\mathbf{r}_1)-(\mathbf{r}_2-2\mathbf{r}_1) = \mathbf{r}_1-\mathbf{r}_2+\mathbf{r}_3 = (0,0,0,1)$, so for $\mathbf{y} = (1,-1,1)$, we have $\mathbf{y}^T A = \mathbf{0}^T$ (or equivalently, $A^T\mathbf{y} = \mathbf{0}$), and $\mathbf{y}^T\mathbf{b} = 1$.
In general, if $A\mathbf{x} = \mathbf{b}$ has no solution, then taking the reduced row echelon form of the whole augmented matrix $[A|\mathbf{b}]$, we get $[EA\,|\,E\mathbf{b}]$ (E being the product of the row operation matrices), where the last nonzero row is $[\mathbf{0}^T|1]$. If this last nonzero row is the ith row then let $\mathbf{e}_i$ be the i'th standard basis element $(0,\dots,0,1,0,\dots,0)$ of $\mathbf{R}^m$. Then $\mathbf{e}_i^T E[A|\mathbf{b}] = [\mathbf{0}^T|1]$, so for the row vector $\mathbf{y}^T = \mathbf{e}_i^T E$, we have $\mathbf{y}^T A = \mathbf{0}^T$ (implying $\mathbf{y} \in N(A^T)$), and $\mathbf{y}^T\mathbf{b} = 1$.
Conversely, if there is such a $\mathbf{y}$, then there cannot exist an $\mathbf{x}$ with $A\mathbf{x} = \mathbf{b}$, because then $\mathbf{y}^T A\mathbf{x} = \mathbf{0}^T\mathbf{x} = 0$ but $\mathbf{y}^T b = 1$.
So one and only one of the two condition always holds.

4. *Determine the projection matrix onto the column space and onto the left nullspace of R_1. Do the same for R_2. Comment on the results.*

$$R_1 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}; \qquad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution: The columns of both R_1 and R_2 are independent, so we can use the formula $A(A^TA)^{-1}A^T$ for the projection matrix on the column space of A.
$R_1^TR_1 = \begin{bmatrix} 21 & 9 \\ 9 & 4 \end{bmatrix}$, and $(R_1^TR_1)^{-1} = \begin{bmatrix} 4/3 & -3 \\ -3 & 7 \end{bmatrix}$. Then the projection matrix on $C(R_1)$ is

$$P_1 = R_1(R_1^TR_1)^{-1}R_1^T = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4/3 & -3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -2 \\ -1/3 & 1 \\ -1/3 & 1 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

From this, the projection matrix on $N(R_1^T)$ (that is, on the orthogonal complement of $C(R_1)$) is

$$I - P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2/3 & -1/3 & -1/3 \\ 0 & -1/3 & 2/3 & -1/3 \\ 0 & -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

The same for R_2 gives $R_2^TR_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $(R_2^TR_2)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$, and the projection matrix on $C(R_2)$

$$P_2 = R_2(R_2^TR_2)^{-1}R_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \\ 0 & 1/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix},$$

and the projection matrix on $N(R_2^T) = C(R_2)^\perp$ is

$$I - P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2/3 & -1/3 & -1/3 \\ 0 & -1/3 & 2/3 & -1/3 \\ 0 & -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

We can see that we got the same projection matrices on $C(R_1)$ and $C(R_2)$ (and then also on $N(R_1^T)$ and $N(R_2)^T$), so the two matrices must have the same column spaces (the column space of the projection matrix is actually the subspace it projects on). Indeed, $(1,0,0,0) = (3,2,2,2) - 2(1,1,1,1)$ and $(0,1,1,1) = 3(1,1,1,1) - (3,2,2,2)$, so $C(R_2) \le C(R_1)$, and conversely, $(3,2,2,2) = 3(1,0,0,0) + 2(0,1,1,1)$ and $(1,1,1,1) = (1,0,0,0) + (0,1,1,1)$, so $C(R_1) \le C(R_2)$.

5. *Let $W \le V = \mathbb{R}^{10}$ have dimension 7.*
(i) What is the dimension of its orthogonal complement $W^\perp$?
(ii) What are the possible dimensions of the subspaces $U \le V$ orthogonal to W?
(iii) Suppose $C(A) = W$. What is the minimal number of columns/rows of A?
(iv) Suppose $N(A) = W$. What is the minimal number of columns/rows of A?

Solution: (i) $\dim W^\perp = \dim V - \dim W = 10 - 7 = 3$.

(ii) A subspace $U \le V$ is orthogonal to W if and only if U is a subspace of $W^\perp$, so $\dim U \le 3$ (and, clearly, there exist subspaces of $W^\perp$ of dimension $0, 1, 2$ or 3).

(iii) Since $W \le \mathbb{R}^{10}$, the columns of A are in $\mathbb{R}^{10}$. These columns span a 7-dimensional subspace, so there must be at least 7 columns. Since the columns are in $\mathbb{R}^{10}$, there must be exactly 10 rows.

(iv) $N(A) = W \le \mathbb{R}^{10}$, so the rows of A are also in $\mathbb{R}^{10}$, which implies that there are exactly 10 columns. On the other hand, $\dim(C(A^T)) = r(A) = 10 - 7 = 3$, so there must be at least 3 rows.

6. **HW** *Let $x + y - 2z = 0$ describe a plane $\mathcal{P}$ in the space $\mathbb{R}^3$. Determine the 1×3 matrix A for which $N(A) = \mathcal{P}$. Find the special solutions $\mathbf{s}_1$, $\mathbf{s}_2$ and a basis for the orthogonal complement $\mathcal{P}^\perp$.*
Split $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ into nullspace and row space components.

7. *Determine the projection matrices onto the column space of R and onto the left nullspace. Answer these questions for R^T, too.*

$$R = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: We can see immediately from the matrix R that its rank is 3 (one more step to a row echelon form), and a basis of $C(R)$ is $\{(2,0,0,0), (0,2,0,0), (4,5,-1,1)\}$ but we can replace this basis with a simpler one by changing it to another spanning set of three elements: $\{(1,0,0,0), (0,1,0,0), (0,0,-1,1)\}$, doing elementary column operations on the matrix. Then we can use the following matrix A for calculating the projection matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$$

The projection matrix on the left nullspace is

$$I - P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

We can find a basis for the row space by doing elementary row operations on R:

$$R = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $\{(1,0,1,0),(0,1,3/2,1),(0,0,0,1)\}$ is a basis of $C(R^T)$. We use the matrix B with these basis elements as column to calculate the projection matrix P'.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B^TB = \begin{bmatrix} 2 & 3/2 & 0 \\ 3/2 & 13/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, (B^TB)^{-1} = \frac{1}{17}\begin{bmatrix} 13 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 17 \end{bmatrix},$$

$$P' = B(B^TB)^{-1}B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{17}\begin{bmatrix} 13 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\frac{1}{17}\begin{bmatrix} 13 & -6 & 0 \\ -6 & 8 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 13 & -6 & 4 & 0 \\ -6 & 8 & 6 & 0 \\ 4 & 6 & 13 & 0 \\ 0 & 0 & 0 & 17 \end{bmatrix},$$

and the projection matrix on $N(R)$ is

$$I - P' = \frac{1}{17}\begin{bmatrix} 4 & 6 & -4 & 0 \\ 6 & 9 & -6 & 0 \\ -4 & -6 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(One can check the projection matrices by letting them act on the columns (in case of P', the rows) of R, and on a spanning set of $N(R^T)$ and $N(R)$, respectively.)