1. *Let $\mathbf{b} \in \mathbb{R}^m$ and all entries of $\mathbf{a} \in \mathbb{R}^m$ are 1. We consider $\mathbf{a}x = \mathbf{b}$ with unique real variable x. Let $\hat{x}$ denote the least squares solution. Then*
(i) $\mathbf{a}^T\mathbf{a}\hat{x} = \mathbf{a}^T\mathbf{b}$ shows that $\hat{x}$ is the average (mean) of the entries of $\mathbf{b}$;
(ii) $\|\mathbf{b} - \mathbf{a}\hat{x}\|^2$ is the variance of the entries of $\mathbf{b}$ (its square root is the standard deviation);
(iii) find the projection matrix P onto the line through $\mathbf{a}$.

Solution: (i) $\mathbf{a}^T\mathbf{a} = m$ and $\mathbf{a}^T\mathbf{b} = \sum b_i$, so the normal equation is $m\hat{x} = \sum_{i=1}^{m} b_i$, and its solution is $\hat{x} = \frac{1}{m}\sum b_i =: \bar{b}$ the average of the b_i's.
(ii) $\|\mathbf{b} - \mathbf{a}\bar{b}\|^2 = \sum_{i=1}^{m}(b_i - \bar{b})^2$ is, indeed, the variance of the b_i's.
(iii) The projection of $\mathbf{b}$ on $\mathbf{a}$ is $\mathbf{a}\hat{x} = \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T\mathbf{b} = \mathbf{a}\frac{1}{m}\mathbf{a}^T\mathbf{b} = \frac{1}{m}J\mathbf{b}$, where $J = \mathbf{a}\mathbf{a}^T$ is the $m \times m$ matrix whose every entry is 1. So $P = \frac{1}{m}J$, that is, the $m \times m$ matrix whose every entry is $\frac{1}{m}$.

2. *True or false?*
(i) If Q is an orthogonal matrix then so is Q^{-1}.
(ii) If an $m \times n$ matrix A has orthogonal columns then $\|A\mathbf{v}\| = \|\mathbf{v}\|$ for every vector $\mathbf{v} \in \mathbb{R}^n$.
(iii) A has three orthogonal columns of length 1, 2 and 3 if and only if A^TA is a diagonal matrix with diagonal entries 1, 4 and 9.
(iv) If an orthogonal matrix is lower triangular then it is diagonal.
(v) The determinant of $I + A$ is $1 + |A|$.

Solution: (i) True. Q is orthogonal if it is a square matrix such that $Q^{-1} = Q^T$, that is, $QQ^T = Q^TQ = I$. But then $Q^{-1}(Q^{-1})^T = Q^{-1}(Q^T)^T = Q^{-1}Q = I$, and $(Q^{-1})^TQ^{-1} = (Q^T)^TQ^{-1} = QQ^{-1} = I$, so Q^{-1} is also orthogonal.
(ii) False. For example $A = 2I$ has orthogonal columns but $\|2I\mathbf{v}\| = \|2\mathbf{v}\| \neq \|\mathbf{v}\|$, when $\mathbf{v} \neq \mathbf{0}$. (But the statement would be true if A had orthonormal columns.)
(iii) True. If the columns are $\mathbf{a}_i$ $(i = 1, 2, 3)$, then the (i, j) element of the 3×3 matrix A^TA is $\mathbf{a}_i \cdot \mathbf{a}_j$, so this matrix is diagonal with diagonal entries $1, 4, 9$ if and only if $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ for $i \neq j$, and $\mathbf{a}_i \cdot \mathbf{a}_i = 1, 4, 9$, when $i = 1, 2, 3$.
(iv) True. An orthogonal matrix is always invertible, and the inverse of a lower triangular matrix is always lower triangular. On the other hand, the transpose of a lower triangular matrix is upper triangular, so these two can only be equal when they are both upper and lower triangular, that is, diagonal. The transpose of a diagonal matrix is also diagonal, so the original matrix is diagonal.
(v) False. For example, for $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $|I + A| = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2$, and $|I| + |A| = 1 + 0 = 1$.

3. *Let $b_1 = 0$, $b_2 = 8$, $b_3 = 8$, $b_4 = 20$ be the measurements at times $t_1 = 0$, $t_2 = 1$, $t_3 = 3$, $t_4 = 4$. Find the best straight line approximating these points on the plane by determining and solving the normal equation $A^TA\hat{\mathbf{x}} = \mathbf{b}$. What is the minimum (total) error?*

Solution: We want to determine a straight line $y = C + Dt$ which is closest to the given points in the sense that for $y_i = C + Dt_i$ $(i = 1, 2, 3, 4)$, the value of $\sum_{i=1}^{4}(b_i - y_i)^2$ is minimal. That is, we want to find the optimal approximate solution for unknowns C, D of the system of equations $C + Dt_i = b_i$ $(i = 1, 2, 3, 4)$.
The augmented matrix of the corresponding system of equations is

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 0 & | & 0 \\ 1 & 1 & | & 8 \\ 1 & 3 & | & 8 \\ 1 & 4 & | & 20 \end{bmatrix},$$

and the augmented matrix of the normal equation is

$$[A^TA|A^T\mathbf{b}] = A^T[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & | & 0 \\ 1 & 1 & | & 8 \\ 1 & 3 & | & 8 \\ 1 & 4 & | & 20 \end{bmatrix} = \begin{bmatrix} 4 & 8 & | & 36 \\ 8 & 26 & | & 112 \end{bmatrix}$$

The solution of the latter is

$$\begin{bmatrix} 4 & 8 & | & 36 \\ 8 & 26 & | & 112 \end{bmatrix} \mapsto \begin{bmatrix} 4 & 8 & | & 36 \\ 0 & 10 & | & 40 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & | & 9 \\ 0 & 1 & | & 4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 4 \end{bmatrix} \quad \Rightarrow \hat{C} = 1,\ \hat{D} = 4.$$

So the straight line is $y = 1 + 4t$, and the values taken by this linear function at the given places are $(y_1, y_2, y_3, y_4) = (1, 5, 13, 17)$ instead of $(b_1, b_2, b_3, b_4) = (0, 8, 8, 20)$.
The error is $\|(0, 8, 8, 20) - (1, 5, 13, 17)\| = \|(-1, 3, -5, 3)\| = \sqrt{44}$.

4. *Let* $\mathbf{e}$ *be the error vector for a projection problem. Which is* $\mathbf{e}$ *perpendicular to:* $\mathbf{b}$, $\hat{\mathbf{x}}$, $\mathbf{e}$, $\mathbf{p}$*? Show that* $\|\mathbf{e}\|^2 = \mathbf{e}^T\mathbf{b} = \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\mathbf{p}$.

Solution: $\mathbf{e}$ is perpendicular to the subspace where we project the vectors, so $\mathbf{e}$ is perpendicular to $\mathbf{p}$.
$\mathbf{e} = \mathbf{b} - \mathbf{p}$, so $\mathbf{e}^T\mathbf{e} = \mathbf{e}^T(\mathbf{b} - \mathbf{p}) = \mathbf{e}^T\mathbf{b} - \mathbf{e}^T\mathbf{p} = \mathbf{e}^T\mathbf{b} - 0 = \mathbf{e}^T\mathbf{b}$, and further, $\mathbf{e}^T\mathbf{b} = (\mathbf{b} - \mathbf{p})^T\mathbf{b} = (\mathbf{b}^T - \mathbf{p}^T)\mathbf{b} = \mathbf{b}^T\mathbf{b} - \mathbf{p}^T\mathbf{b}$. (The last term can be written as $\mathbf{p}^T\mathbf{b}$, since the dot product is symmetric.)

5. *Perform the Gram Schmidt orthogonalisation algorithm for the vectors* $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. *Do the same for the columns of* $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Solution:

$$\mathbf{b}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2' = \mathbf{a}_2 - \frac{\mathbf{b}_1\mathbf{a}_2}{\|\mathbf{b}_1\|^2}\mathbf{b}_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -5/2 \end{bmatrix} \,||\, \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{b}_2$$

So we get $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as an orthogonal system, and $\mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as an orthonormal system.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

$$\mathbf{b}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2' = \mathbf{a}_2 - \frac{\mathbf{b}_1\mathbf{a}_2}{\|\mathbf{b}_1\|^2}\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} \,||\, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} = \mathbf{b}_2$$

$$\mathbf{b}_3' = \mathbf{a}_3 - \frac{\mathbf{b}_1\mathbf{a}_3}{\|\mathbf{b}_1\|^2}\mathbf{b}_1 - \frac{\mathbf{b}_2\mathbf{a}_3}{\|\mathbf{b}_2\|^2}\mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{-2}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix} \,||\, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} = \mathbf{b}_3$$

So $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ is an orthogonal system such that $\operatorname{span}(\mathbf{b}_1, \ldots, \mathbf{b}_i) = \operatorname{span}(\mathbf{a}_1, \ldots, \mathbf{a}_i)$ for $i = 1, 2, 3$, and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ is the corresponding orthonormal system:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \ \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \ \mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{q}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \ \mathbf{q}_3 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$$

6. HW *Let* $x + y - 2z = 0$ *describe a plane* $\mathcal{P}$ *in the space* $\mathbb{R}^3$*. Find two orthogonal vectors in* $\mathcal{P}$ *and make them orthonormal. Extend it to an orthonormal basis of* $\mathbb{R}^3$.

7. *Determine the following determinants:*

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{vmatrix}, \qquad \begin{vmatrix} 1 & a & a^2 \\ a & 1 & a \\ a^2 & a & 1 \end{vmatrix}, \qquad \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}, \qquad \det\left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right).$$

Solution:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{vmatrix} = -3 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & -3 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 3 \cdot 1 \cdot 1 \cdot 1 = 3.$$

$$\begin{vmatrix} 1 & a & a^2 \\ a & 1 & a \\ a^2 & a & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a^2 & a-a^3 \\ 0 & a-a^3 & 1-a^4 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a^2 & a-a^3 \\ 0 & 0 & 1-a^2 \end{vmatrix} = (1-a^2)^2 = a^4 - 2a^2 + 1.$$

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{vmatrix} = 5.$$

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \det \begin{bmatrix} 32 \end{bmatrix} = 32.$$

8. *Find the determinant of rotations and reflections:*

$$Q_1 = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} \cos 2\vartheta & -\sin 2\vartheta \\ -\sin 2\vartheta & -\cos 2\vartheta \end{bmatrix}.$$

Solution: $|Q_1| = \cos^2\vartheta + \sin^2\vartheta = 1$, and $|Q_2| = -\cos^2 2\vartheta - \sin^2 2\vartheta = -1$.

9. *Observe that* 7 *divides* 343, 147 *and* 504*. Surprise surprise,* 7 *divides the following determinant. Why?*

$$\begin{bmatrix} 3 & 4 & 3 \\ 1 & 4 & 7 \\ 5 & 0 & 4 \end{bmatrix}$$

Solution: The determinant is the same if we take the transpose of the matrix. Then we can add 100 times the first row and 10 times the second row to the third, without changing the determinant.

$$\begin{vmatrix} 3 & 4 & 3 \\ 1 & 4 & 7 \\ 5 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 5 \\ 4 & 4 & 0 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 5 \\ 4 & 4 & 0 \\ 343 & 147 & 504 \end{vmatrix} = 7 \cdot \begin{vmatrix} 3 & 1 & 5 \\ 4 & 4 & 0 \\ 49 & 21 & 72 \end{vmatrix}.$$

But it follows from the multilinearity and the determinant of the permutation matrices (in fact from the Big formula) that the determinant of a matrix with integer entries is also an integer, so the determinant of the original matrix is divisible by 7.

10. *What is the determinant of a projection matrix?*

Solution: The rank of the matrix of the projection on a subspace $V \leq \mathbb{R}^n$ is the dimension of V. So if V is not the whole $\mathbb{R}^n$, then a row echelon form of the matrix has a zero row, which proves that the determinant is 0. (If $V = \mathbb{R}^n$, then the projection matrix is I, so its determinant is 1 in this case.)