1. *Find the nonzero terms in the big formula of* $\det A$ *(out of the 24). What are the signs?*

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 0 \\ 5 & 6 & 0 & 7 \\ 8 & 0 & 0 & 9 \end{bmatrix}$$

Solution: We have to pick a nonzero element from each row and column. From column 3 we can only choose 4 and then from column 2 only 6 because we cannot pick two elements from the same row. Now we have two choices for column 1 and 4: either we take 1 and 9 or 8 and 2. So $\det A$ is the sum of two terms: $|A| = |P_1| \cdot 1 \cdot 4 \cdot 6 \cdot 9 + |P_2| \cdot 2 \cdot 4 \cdot 6 \cdot 8$. The permutation matrix P_1 which belongs to the permutation 1324 has determinant $(-1)^1 = 1$, since this permutation can be brought to 1234 by one swap only (this means one row exchange in P_1 to bring it to I), and P_2 belongs to the permutation 4321, and here we need two swaps, so $|P_2| = (-1)^2 = 1$. This gives $|A| = -1 \cdot 4 \cdot 6 \cdot 9 + 2 \cdot 4 \cdot 6 \cdot 8 = 24(-9+16) = 24 \cdot 7 = 168$.

2. *True or false?*
(i) If the main diagonal entries of a matrix are all 0 *then the determinant is* 0.
(ii) If $|A| = 0 = |B|$ *then* $|A+B| = 0$.
(iii) If $|A| = 0 = |B|$ *then* $|AB| = 0$.
(iv) If $|A| = 0$ *then there is an entry of* A *which is* 0.
(v) If $|A| > 0$ *then there is an entry of* A *which is positive.*

Solution: (i) False. For example $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1 \neq 0$.

(ii) False. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have $|A| = |B| = 0$ but $|A+B| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$.

(iii) True. If A and B are $n \times n$ matrices then $|A| = |B| = 0$ implies that $r(A), r(B) < n$, so $r(AB) \leq \min\{r(A), r(B)\} < n$, thus $|AB| = 0$.

(iv) False. For example $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$.

(v) False. For example $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$.

3. *Write recursive formulas for* $a_n = |A_n|$, $b_n = |B_n|$ *and* $c_n = |C_n|$ *and determine their values.*

$$A_n = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}; \; B_n = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}; \; C_n = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 1 & -1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 1 & -1 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Solution: If we use the cofactor formula for the first row, and in calculating the second cofactor, we expand the determinant along the first column then we get $a_n = 2a_{n-1} - (-1)(-1)a_{n-2} = 2a_{n-1} - a_{n-2}$. Starting with $A_1 = \det[2] = 2$, and $a_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3$, and calculating a few more values by the recursive formula, we get $a_3 = 4$, $a_4 = 5$, $a_5 = 6$, which gives the conjecture that $a_n = n+1$. Indeed, if this is true up to n, then also true for $n+1$: $a_{n+1} = 2a_n - a_{n-1} = 2(n+1) - n = n+2 = (n+1)+1$. So it is correct for every n.
(Or: we can rewrite the recursion as $a_n - a_{n-1} = a_{n-1} - a_{n-2}$, and using this recursively, $a_n - a_{n-1} = a_{n-1} - a_{n-2} = a_{n-2} - a_{n-3} = \ldots = a_2 - a_1 = 3 - 2 = 1$, so $a_n = a_{n-1} + 1 = a_{n-2} + 2 = \ldots = a_1 + (n-1) = 2 + (n-1) = n+1$.)
If we start reducing the determinant to smaller determinants in case of B_n the same way as in the case of A_n, then we get $b_n = a_{n-1} - a_{n-2} = n - (n-1) = 1$ for $n \geq 3$, while $b_1 = 1$ and $b_2 = 2 - (-1)^2 = 1$ also holds.

But we can also find a recursion for b_n using only b_k's if we start expanding along the last row, and then in the second term along the last column: $b_n = 2b_{n-1} - b_n$, the same recursion as before, only here the initial values are different. Now we would get $b_n - b_{n-1} = b_{n-1} - b_{n-2} = \ldots = b_2 - b_1 = 1 - 1 = 0$, so $b_n = b_{n-1} = \ldots = b_1 = 1$.
Finally, starting to expand the determinant of C_n the same way as for A_n, we get the recursion $c_n = c_{n-1} + c_{n-2}$, which is the well-known recursion defining the Fibonacci sequence, and the initial members of the sequence here are $c_1 = 1$ and $c_2 = 2$, so we get, indeed, the Fibonacci numbers (in fact, the usual indexing there is different, $f_0 = 0$, $f_1 = f_2 = 1$, $f_3 = 2$, …, so $c_n = f_{n+1}$.

4. *Find all the cofactors of A_3 of the previous exercise. Form the cofactor matrix, the adjugate matrix and A_3^{-1}.*
Solution:

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Then $|A_3| = 8 + 0 + 0 - 0 - 2 - 2 = 4$, so

$$\operatorname{adj} A_3 = C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and } A_3^{-1} = \frac{1}{|A_3|} \operatorname{adj} A_3 = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

5. *Use Cramer's Rule to solve* $\begin{array}{rcrcl} x & + & 2y & = & 1 \\ 2x & + & y & = & 3 \end{array}.$

Solution:

$$x = \frac{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{5}{3}, \qquad y = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} = -\frac{1}{3}$$

6. *a) Determine the volume of the tetrahedron (inscribed into the unit cube) with vertices:* $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$.
b) **HW** *Let $P = (2,1)$, $Q = (7,2)$, $R = (-1,8)$ and $S = (0,9)$. Determine the area of the triangle PQR and of the quadrangle $PQSR$.*

Solution: The signed volume is $\frac{1}{6}$ times the determinant of the matrix whose rows are the edge vectors from one vertex of the tetrahedron. Here it is most convenient to take the edges starting at $(0,0,0)$.

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0 + 0 + 0 - 0 - 1 - 1 = -2,$$

so the volume is $|\frac{1}{6} \cdot (-2)| = \frac{1}{3}$.

7. *Argue that the area of a triangle that has integer lattice point coordinates is an integer multiple of $1/2$, and the volume of a tetrahedron that has integer lattice point coordinates is an integer multiple of $1/6$.*

Solution: The area of the triangle is $\frac{1}{2}$ times the absolute value of the determinant of the 2×2 matrix, whose rows are the edge vectors of the triangle starting at one vertex. Since this is a determinant of a matrix with integer entries, it follows from the big formula that the determinant is an integer. In the case of the tetrahedron, we get similarly that the volume is $\frac{1}{6}$ times the absolute value of the determinant of a 3×3 matrix with integer entries (the rows are the edge vectors starting from one vertex), so here we get an integer multiple of $\frac{1}{6}$.

8. *Suppose A has orthonormal columns. Using A^TA justify that* $\det(A) = \pm 1$.

Solution: Only square matrices have determinants, so it is assumed in the problem that A is an $n \times n$ matrix. Since its columns are orthonormal (that is, A is an orthogonal matrix), $A^TA = I$, so by the product rule $1 = |I| = |A| \cdot |A^T| = |A|^2$, and this implies that $|A| = \pm 1$.

9. *Rearragning the three vectors, which of the triple products* $(\ \times\ )\cdot$ *are the same as* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$?

Solution: We obtain the triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ as the determinant of the 3×3 matrix whose rows are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in this order. If we switch two vectors then the determinant changes its sign. So we get the same triple product if and only if we use an even number of swaps on the vectors. Thus the triple product is the same for $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$ and $(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$ (these need two swaps), and has opposite sign in the other three cases. That is,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = -(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}.$$

10. *A point $\mathbf{u}$ in space is the position of the mass on which a force $\mathbf{F}$ acts (also a 3-dimensional vector). Observe that if $\mathbf{u}$ and $\mathbf{F}$ are parallel then there is no turning. However if they are not parallel then there is. The rotational force, the torque is exactly $\mathbf{u} \times \mathbf{F}$. Check that the direction if correct.*

Solution: If $\mathbf{u}$ and $\mathbf{F}$ are parallel then $\mathbf{u} \times \mathbf{F} = \mathbf{0}$, so the torque is $\mathbf{0}$. Otherwise $\mathbf{u} \times \mathbf{F}$ is perpendicular to the plane of $\mathbf{u}$ and $\mathbf{F}$, and the direction indicates which way the mass is turning with respect to an axis through the origin: if we look at the plane from the end point of $\mathbf{u} \times \mathbf{F}$ then the turn is counterclockwise.

11. *Given a matrix your friend picks a row. You win if you can change at most one element of that row to make the determinant 0. Can you always win?*

Solution: If every cofactor corresponding to the entries of the given row is zero then the determinant is already zero, so I win. If there is a nonzero cofactor whose value is $c \neq 0$, and the determinant of the whole matrix is d, then adding $-d/c$ to the corresponding element, the determinant becomes 0, since $(-d/c) \cdot c$ is added to it when we expand the determinant along the given row. So I win in that case, as well.

12. *Let* $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$. *Determine the eigenvalues and corresponding eigenvectors of A and of $A + I$.*

Solution: $\begin{vmatrix} 1-\lambda & -2 \\ -3 & 6-\lambda \end{vmatrix} = \lambda^2 - 7\lambda = \lambda(\lambda - 7) = 0$ if $\lambda = 0$ or $\lambda = 7$. The eigenvectors for $\lambda = 0$ are the solutions of $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The eigenvectors for $\lambda = 7$ are the solutions of $(A - 7I)\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}.$$

$$A + I = \begin{bmatrix} 2 & -2 \\ -3 & 7 \end{bmatrix}, \qquad |(A+I) - \lambda I| = \begin{vmatrix} 2-\lambda & -2 \\ -3 & 7-\lambda \end{vmatrix} = \lambda^2 - 9\lambda + 8 = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 8$$

The eigenvectors for $\lambda_1 = 1$ are in the nullspace of $\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$, and for $\lambda_2 = 8$ in the nullspace of $\begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix}$, the same as in the case of A, so the eigenvectors are also the same: $x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for λ_1 and $x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ for λ_2.

13. *Determine the eigenvalues and corresponding eigenvectors of* $B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$*, of* B^2 *and of* B^{-1}*.*

Solution:

$$|B - \lambda I| = \begin{bmatrix} 1-\lambda & 3 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 6 = 0 \;\Rightarrow\; \lambda_1 = 3, \quad \lambda_2 = -2$$

Eigenvectors:

$$\text{for } \lambda_1 = 3: \quad \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \;\Rightarrow\; \mathbf{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix},$$

$$\text{for } \lambda_2 = -2: \quad \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \;\Rightarrow\; \mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$B^2 = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}, \quad |B^2 - \lambda I| = \begin{vmatrix} 7-\lambda & 3 \\ 2 & 6-\lambda \end{vmatrix} = \lambda^2 - 13\lambda + 36 = 0 \;\Rightarrow\; \lambda_1 = 9, \quad \lambda_2 = 4$$

Eigenvectors:

$$\text{for } \lambda_1 = 9: \quad \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \mathbf{x} = \mathbf{0} \;\Leftrightarrow\; \mathbf{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix},$$

$$\text{for } \lambda_2 = 4: \quad \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0} \;\Leftrightarrow\; \mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$B^{-1} = \begin{bmatrix} 0 & 1/2 \\ 1/3 & -1/6 \end{bmatrix}, \quad |B^{-1} - I| = \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{6} - \lambda \end{vmatrix} = \lambda^2 + \frac{1}{6}\lambda - \frac{1}{6} = 0 \;\Rightarrow\; \lambda_1 = \frac{1}{3}, \quad \lambda_2 = -\frac{1}{2}$$

The eigenvectors:

$$\text{for } \lambda_1 = \frac{1}{3}: \quad \begin{bmatrix} -1/3 & 1/2 \\ 1/3 & -1/2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \;\Rightarrow\; \mathbf{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$$\text{for } \lambda_2 = -\frac{1}{2}: \quad \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 1/3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \;\Rightarrow\; \mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

14. *Explain the general phenomena governing the previous two exercises.*

Solution: If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, and $B = f(A) = c_m A^m + c_{m-1} A^{m-1} + \ldots + c_1 A + c_0 I$ is a polynomial of A, then $\mathbf{v}$ is also and eigenvector of B with eigenvalue $\mu = f(\lambda) = c_m \lambda^m + c_{m-1}\lambda^{m-1} + \ldots + c_1\lambda + c_0$. Indeed, $A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda\lambda\mathbf{v} = \lambda^2\mathbf{v}$, and for higher powers $A^k\mathbf{v} = A^{k-1}\lambda\mathbf{v} = A^{k-2}\lambda^2\mathbf{v} = \ldots = A\lambda^{k-1}\mathbf{v} = \lambda^k\mathbf{v}$, so $f(A)\mathbf{v} = (\sum_k c_k A^k)\mathbf{v} = \sum_k (c_k A^k \mathbf{v}) = \sum_k c_k \lambda^k \mathbf{v} = f(\lambda)\mathbf{v}$.

Furthermore, if A is invertible, and $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then λ cannot be zero (because $N(A) = \{\, \mathbf{0} \,\}$ by the invertibility of A), and $A\mathbf{v} = \lambda\mathbf{v}$ implies that $\mathbf{v} = I\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v}$, so $A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$, which means that $\mathbf{v}$ is also and eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.