1. *Determine the polar form of* $1-3i$. *What is* $(1/(1-3i))^2+(1/(1+3i))^2$*?*

Solution: $z = r(\cos\varphi + i\sin\varphi)$, where $r = |1-3i| = \sqrt{10}$ and $\varphi = \tan^{-1}(-3) = -\tan^{-1}3$, since $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ and $\tan\varphi = \frac{\mathrm{Im}z}{\mathrm{Re}z} = -3$.
$1/(1-3i) = (1+3i)/(1-3i)(1+3i) = (1+3i)/10$ and $1/(1+3i) = (1-3i)/(1+3i)(1-3i) = (1-3i)/10$, so $(1/(1-3i))^2+(1/(1+3i))^2 = \frac{1}{100}((1+3i)^2+(1-3i)^2) = \frac{1}{100}(1-9+6i+1-9-6i) = -\frac{4}{25}$.

2. *Let* $z, w \in \mathbb{C}$ *such that* $|z| = 5$, $|w| = 3$. *What is* $|zw|$, $|z/w|$, $|z+w|$, $|z-w|$*? If some cannot be determined exactly then give lower and upper bounds for them.*

Solution: $|zw| = 5\cdot 3 = 15$, $|z/w| = 5/3$, $||z|-|w|| \le |z+w| \le |z|+|w|$, so in this case $2 \le |z+w| \le 8$, and applying the same for $-w$ instead of w we get $2 \le |z-w| \le 8$.

3. *True or false?*
(i) Every complex number has a square root.
(ii) Every comples number has a 100-th root.
(iii) If a complex number is not real then its square root is not real.
(iv) If a complex number is purely imaginary then its square root is purely imaginary.
(v) The square roots of a negative real number are purely imaginary.

Solution: (i) and (ii) are both true.
In general, if $z = r(\cos\varphi + i\sin\varphi)$ then $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos(\frac{\varphi}{n} + k\frac{2\pi}{n}) + i\sin(\frac{\varphi}{n} + k\frac{2\pi}{n})\right)$ are the nth roots of z, or if φ is given in degrees then $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos(\frac{\varphi}{n} + k\frac{360^\circ}{n}) + i\sin(\frac{\varphi}{n} + k\frac{360^\circ}{n})\right)$, where $k = 0, 1, \ldots, (n-1)$.
(iii) True. The original number is the square of any of its square roots and the square of a real number is real.
(iv) False. $(1+i)^2 = 2i$ is purely imaginary but $1+i$ is not.
(v) True. If a is a positive real number then $\sqrt{-a} = \pm i\sqrt{a}$.

4. *Find the eigenvalues and eigenvectors of the following rank 1 matrices.*

$$A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -2 & 0 \\ 6 & -4 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ \pi \\ -\pi \end{bmatrix} \begin{bmatrix} \sqrt{2} & -1 & -1 \end{bmatrix}; \quad C = \frac{\mathbf{x}\mathbf{y}^T}{\mathbf{x}^T\mathbf{y}}$$

Solution: An $n \times n$ rank 1 matrix has a 1-dimensional column space, say, $\mathrm{span}(\mathbf{a})$, and then $A\mathbf{v} \in \mathrm{span}(\mathbf{a})$ can only be $\lambda\mathbf{v}$ if either $\mathbf{v}$ itself is in $\mathrm{span}(\mathbf{a})$, or $\lambda = 0$. In the latter case $\mathbf{v}$ is in the $(n-1)$-dimensional nullspace of the matrix.
For A, $\mathcal{C}(A) = \mathrm{span}((0,1,2))$, thus the (nonzero) scalar multiples of $(0,1,2)$ are eigenvectors, and since

$$A\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix} = -2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

the nonzero scalar multiples of $(0,1,2)$ are eigenvectors with eigenvalue -2, while the eigenvectors with eigenvalue 0 are the nonzero vectors of the subspace spanned by the special solutions of $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & -2 & 0 \\ 6 & -4 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{s}_1 = \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In general we can say that if for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ nonzero vectors, $M = \mathbf{a}\mathbf{b}^T$, then the eigenvectors of M are the nonzero scalar multiples of $\mathbf{a}$ with eigenvalue $\mathbf{b}^T\mathbf{a} = \mathbf{a}\cdot\mathbf{b}$, since $\mathbf{a}\mathbf{b}^T\mathbf{a} = (\mathbf{b}^T\mathbf{a})\mathbf{a}$, and the nonzero vectors of $(\mathrm{span}(\mathbf{b}))^\perp$, since the latter is $(n-1)$-dimensional, and for any $\mathbf{v}$ in this subspace $\mathbf{a}\mathbf{b}^T\mathbf{v} = \mathbf{a}0 = \mathbf{0}$.
By the general rule above, the eigenvalues of B are $\sqrt{2}$ and 0, the eigenvectors for $\sqrt{2}$ are the nonzero scalar multiples of $\begin{bmatrix} 1 \\ \pi \\ -\pi \end{bmatrix}$, and the eigenvectors for 0 are the nozero vectors of

$$\mathrm{span}((\sqrt{2}, -1, -1))^\perp = \mathrm{span}((1, \sqrt{2}, 0), (1, 0, \sqrt{2})).$$

$C = \frac{1}{\mathbf{x}^T\mathbf{y}}\mathbf{x}\mathbf{y}^T$, so the eigenvalues are $\left(\frac{1}{\mathbf{x}^T\mathbf{y}}\mathbf{x}\right)\cdot\mathbf{y} = 1$ and 0, and the eigenvectors are in span($\mathbf{x}$) for 1, and in $(\text{span}(\mathbf{y}))^\perp$ for 0.

5. *Simon says: "every triangular matrix has its eigenvalues on the diagonal." Is he right?*

Solution: Yes. If A is triangular with diagonal elements $d_1, \dots, d_n$, then $A - \lambda I$ is also diagonal with diagonal elements $d_1 - \lambda,\ d_2 - \lambda, \dots,\ d_n - \lambda$, so the eigenvalues are the roots of $|A - \lambda I| = (d_1 - \lambda)\cdots(d_n - \lambda)$.

6. HW *Find four independent eigenvectors of D and thus diagonalise it.*

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

7. *Find the eigenvalues of the following permutation matrix.*

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution:

$$|P - \lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{vmatrix} = (-\lambda)^4 - 1^4 = \lambda^4 - 1,$$

so the eigenvalues are the fourth roots of 1: ± 1 and $\pm i$.

8. *Suppose $AX = X\Lambda$ with X invertible. Which is true?*
(i) If X is triangular then A is triangular.
(i) If A is triangular then X is triangular.
(i) The columns of $(X^{-1})^T$ are eigenvectors of A^T.

Solution: (i) True. Suppose X is lower triangular (for upper triangular the same arguments work). If we multiply the equation by X^{-1} from the left, we see that $A = X\Lambda X^{-1}$, where X and Λ are both lower triangular (Λ is even diagonal), and then X^{-1} and $A = X\Lambda X^{-1}$ are also lower triangular (see problem 4/9).

(ii) False $A = I$ is triangular, and $IX = XI$ for every matrix X, not only for triangular matrices.

(iii) True. $AX = X\Lambda \Rightarrow$ (by multiplying the equation by X^{-1} from right and left) $X^{-1}A = \Lambda X^{-1}$ $\Rightarrow$ (by transposing) $A^T(X^{-1})^T = (X^{-1})^T\Lambda^T = (X^{-1})^T\Lambda$, so Λ is also the diagonal form of A^T, and the columns of the also invertible matrix $(X^{-1})^T$ are corresponding eigenvectors.

9. *Let $g_0 = 0$ and $g_1 = 1$. For $n > 1$ define $g_n = \frac{g_{n-1} + g_{n-2}}{2}$. (This sequence of consecutive means is sometimes called "Gibonacci" series.) Using matrices determine a formula for g_n, and show that $g_n \to 2/3$.*

Solution:

$$\begin{bmatrix} g_n \\ g_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}g_{n-1} + \frac{1}{2}g_{n-2} \\ g_{n-1} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} g_{n-1} \\ g_{n-2} \end{bmatrix},$$

so for

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} g_n \\ g_{n-1} \end{bmatrix} = A^{n-1} \begin{bmatrix} g_1 \\ g_0 \end{bmatrix} = A^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

First we diagonalize the matrix A for calculating its powers.

$$|A - \lambda I| = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} \quad \Rightarrow \lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}.$$

Eigenvectors for

$$\lambda_1 = 1: \quad x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -\frac{1}{2}: \quad x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix},$$

so we can use the eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ for diagonalization.

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad X^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \qquad X^{-1}AX = \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

$$A^n = (X\Lambda X^{-1})^n = X\Lambda^n X^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-1/2)^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} =$$

$$\frac{1}{3}\begin{bmatrix} 2+(-1/2)^n & 1-(-1/2)^n \\ 2+(-1/2)^{n-1} & 1-(-1/2)^{n-1} \end{bmatrix}, \quad \Rightarrow \quad \begin{bmatrix} g_n \\ g_{n-1} \end{bmatrix} = A^{n-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2+(-1/2)^{n-1} \\ 2+(-1/2)^{n-2} \end{bmatrix}$$

So $g_n = \frac{1}{3}(2+(-1/2)^{n-1}) \to \frac{2}{3}$.

10. *Diagonalise A to $X^{-1}AX = \Lambda$ and determine the limits $\Lambda^n \to \Lambda^\infty$ and $A^n \to A^\infty$.*

$$A = \begin{bmatrix} 0 & 0.2 & 0.4 \\ 0.3 & 0.4 & 0.5 \\ 0.7 & 0.4 & 0.1 \end{bmatrix}$$

Solution:

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 0.2 & 0.4 \\ 0.3 & 0.4-\lambda & 0.5 \\ 0.7 & 0.4 & 0.1-\lambda \end{vmatrix} = -\lambda(\lambda^2-0.5\lambda-0.16) - 0.2(-0.3\lambda-0.32) + 0.4(0.7\lambda-0.16) =$$

$$= -\lambda^3 + 0.5\lambda^2 + 0.5\lambda = -\lambda(\lambda^2 - 0.5\lambda - 0.5) \quad \Rightarrow \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = -0.5.$$

Find an eigenvector for each of the three eigenvalues (they will be independent).

$$\lambda_1 = 1\colon\ A-I = \begin{bmatrix} -1 & 0.2 & 0.4 \\ 0.3 & -0.6 & 0.5 \\ 0.7 & 0.4 & -0.9 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -0.2 & -0.4 \\ 0 & -0.54 & 0.62 \\ 0 & 0.54 & -0.62 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -\frac{17}{27} \\ 0 & 1 & -\frac{31}{27} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 17 \\ 31 \\ 27 \end{bmatrix}$$

$$\lambda_2 = 0\colon\ A = \begin{bmatrix} 0 & 0.2 & 0.4 \\ 0.3 & 0.4 & 0.5 \\ 0.7 & 0.4 & 0.1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 2 \\ 1 & \frac{4}{7} & \frac{1}{7} \end{bmatrix} \mapsto\mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -0.5\colon\ A-0.5I = \begin{bmatrix} 0.5 & 0.2 & 0.4 \\ 0.3 & 0.9 & 0.5 \\ 0.7 & 0.4 & 0.6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0.4 & 0.8 \\ 0 & 0.78 & 0.26 \\ 0 & 0.12 & 0.04 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 17 & 1 & -2 \\ 31 & -2 & -1 \\ 27 & 1 & 3 \end{bmatrix}, \qquad X^{-1} = \frac{1}{75}\begin{bmatrix} 1 & 1 & 1 \\ 24 & -21 & 9 \\ -17 & -2 & 13 \end{bmatrix}, \qquad X^{-1}AX = \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}$$

$$A^\infty = X\Lambda^\infty X^{-1} = \begin{bmatrix} 17 & 1 & -2 \\ 31 & -2 & -1 \\ 27 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{75} \begin{bmatrix} 1 & 1 & 1 \\ 24 & -21 & 9 \\ -17 & -2 & 13 \end{bmatrix} = \frac{1}{75}\begin{bmatrix} 17 & 17 & 17 \\ 31 & 31 & 31 \\ 27 & 27 & 27 \end{bmatrix}.$$

(Actually, less would have been enough for calculating A^∞. If the rows of X^{-1} are $\mathbf{w}_1^T, \ldots, \mathbf{w}_n^T$ (left eigenvectors for $\lambda_1, \ldots, \lambda_n$), and the columns of X are $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (right eigenvectors for $\lambda_1, \ldots, \lambda_n$), then $A^m = X\Lambda^m X^{-1} = [\lambda_1^m \mathbf{v}_1 \ \ldots \ \lambda_n^m \mathbf{v}_n]X^{-1} = \lambda_1^m \mathbf{v}_1 \mathbf{w}_1^T + \ldots + \lambda_n^m \mathbf{v}_n \mathbf{w}_n^T$, and since here $\lambda_i^m \to 0$ for $i = 2, 3$, $\Lambda^\infty = \mathrm{diag}(1,0,0)$, we have $A^\infty = \mathbf{v}_1 \mathbf{w}^T$, that is, we did not need the eigenvectors for the small eigenvalues.)