1. *Determine the cubic roots of* $-2+2i$.

Solution: $z=-2+2i=\sqrt{8}(\cos 135^\circ + i\sin 135^\circ)$
$\sqrt[3]{z}=\sqrt{2}(\cos(45^\circ+k\cdot 120^\circ)+i\sin(45^\circ+k\cdot 120^\circ))$, $k=0,1,2$, that is, the absolute value of the cubic roots is $\sqrt{2}$, and the angles are 45°, 165° and 285°.

2. *Find all complex numbers such that* $z^2+(2i-2)z-2-2i=0$.

Solution:

$$z=\frac{2-2i\pm\sqrt{(-2+2i)^2-4(-2-2i)}}{2}=\frac{2-2i\pm\sqrt{-8i+8+8i}}{2}=(1\pm\sqrt{2})-i.$$

3. *True or false?*
(i) If A is similar to I then $A=I$.
(ii) If A is similar to $2A$ then $A=0$.
(iii) There are infinitely many regular 2×2 matrices A similar to A^{-1}.

Solution: (i) True. If $X^{-1}AX=I$, then $A=XIX^{-1}=XX^{-1}=I$.

(ii) False. $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, and for $X=\begin{bmatrix}2&0\\0&1\end{bmatrix}$ we get $XAX^{-1}=\begin{bmatrix}0&2\\0&0\end{bmatrix}=2A$, so A and $2A$ are similar. (Actually, if $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A, then $2\lambda_1,\dots,2\lambda_n$ are the eigenvalues of $2A$, so they cannot be similar unless all eigenvalues are 0. This makes the search for a counterexample easier.)

(iii) True. Actually, there are infinitely many matrices which are not only similar but also equal to their own inverses. Every reflection has this property, and there are infinitely many different lines going through the origin, so we get infinitely many such matrices.

4. *Connect similar matrices (only).*

$$A=\begin{bmatrix}0&0&0\\3&-2&0\\6&-4&0\end{bmatrix};\ B=\begin{bmatrix}0&0&0\\3&2&0\\6&4&0\end{bmatrix};\ C=\begin{bmatrix}0&0&0\\0&-2&0\\0&0&0\end{bmatrix};\ D=\begin{bmatrix}-2&0&0\\0&0&0\\0&0&0\end{bmatrix};\ E=\begin{bmatrix}-\lambda&0&0\\3&-2-\lambda&0\\6&-4&-\lambda\end{bmatrix}.$$

Solution: Similar matrices have equal traces, so B is not similar to any of A,C,D. The matrix E is invertible if $\lambda\neq 0,-2$, so in that case it cannot be similar to any of the rank 1 matrices A,B,C,D. For $\lambda=0$, $E=A$, and for $\lambda=-2$, $\operatorname{tr}E=4$ is not equal to the traces of A,B,C,D, so it is not similar to either of them.
We only have to decide if any of A,C,D are similar. Two of these are diagonal, and A is also diagonalizable ($\dim N(A)=2$, so there are two independent eigenvectors for $\lambda=0$, and -2 is also an eigenvalue). But then it follows, that in all of the three cases there are independent eigenvectors for $-2,0,0$, so they are similar to the diagonal matrix D, using these eigenvectors as columns for the conjugating matrix X.
To summarize: A,C,D are similar to each other, B is not similar to any of them, and $E=A$ when $\lambda=0$ but otherwise it is not similar to any of A,B,C,D.

5. *Simon says: "Eigenvectors for $\lambda=0$ span the nullspace and eigenvectors for $\lambda\neq 0$ span the column space." In what is he right, if at all?*

Solution: By definition the nonzero vectors of the nullspace are the eigenvectors for $\lambda=0$, so they, indeed, span the nullspace. It is also true that every eigenvector for an eigenvalue $\lambda\neq 0$ is in the column space: $A\mathbf{v}=\lambda\mathbf{v}\Rightarrow\mathbf{v}=A(\frac{1}{\lambda}\mathbf{v})\in C(A)$. But there may not be enough eigenvectors to span the whole column space. For example, $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ has only one eigenvalue, 1, and the eigenvectors for 1 are only the (nonzero) scalar multiples of $\mathbf{e}_1$, so they do not span the column space, which is $\mathbb{R}^2$.

6. **HW** *We found that a 2×2 real matrix A must have $\operatorname{tr}(A)<0$ and $\det(A)>0$ to be stable (that is, for the solution of $\mathbf{u}'(t)=A\mathbf{u}(t)$ we have $\mathbf{u}(t)\to 0$ if $t\to\infty$). What does it say about*

$$A=\begin{bmatrix}a&b\\c&0\end{bmatrix}?$$

Check the eigenvalues and their real parts.

7. *Find the eigenvalues and the eigenvectors of A to solve the following system of linear differential equations:*

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u}, \text{ that is, for } \mathbf{u}(t) = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}, \quad \begin{array}{rcr} dp/dt & = & 2p+q \\ dq/dt & = & q \end{array}, \quad \text{and } \mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution: The eigenvalues are 2 and 1, and by solving the equations $(A-2I)\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\mathbf{x} = \mathbf{0}$ and $(A-I)\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}$, we get eigenvectors of the form $x_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for the two eigenvalues. Take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Then the solution of the differential equation is

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then $\mathbf{u}(0) = c_1 e^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow c_2 = 1,\ c_1 = 2$, so

$$\mathbf{u}(t) = 2e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} - e^t \\ e^t \end{bmatrix}, \text{ that is, } \begin{array}{rcl} p(t) & = & 2e^{2t} - e^t \\ q(t) & = & e^t. \end{array}$$

8. *Suppose A, B are matrices that are diagonalised by the same X. Show that $AB = BA$.*

Solution: Let $X^{-1}AX = \Lambda_1$ and $X^{-1}BX = \Lambda_2$ diagonal matrices. Clearly, $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ is true for diagonal matrices, so

$$AB = (X\Lambda_1X^{-1})(X\Lambda_2X^{-1}) = X\Lambda_1\Lambda_2X^{-1} = X\Lambda_2\Lambda_1X^{-1} = (X\Lambda_2X^{-1})(X\Lambda_1X^{-1}) = BA.$$

9. *Let $f_n = f_{n-1} - f_{n-2} + f_{n-3}$ be a linear recursion with initial values: $f_1 = 1$, $f_2 = 2$, $f_3 = 3$. Determine f_{100} using the matrix method.*

Solution: The corresponding linear recursion for vectors is

$$\mathbf{F}_n = \begin{bmatrix} f_n \\ f_{n-1} \\ f_{n-2} \end{bmatrix} = \begin{bmatrix} f_{n-1} - f_{n-2} + f_{n-3} \\ f_{n-1} \\ f_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_{n-2} \\ f_{n-3} \end{bmatrix} = A\mathbf{F}_{n-1}.$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 - \lambda + 1 = -(\lambda - 1)(\lambda^2 + 1),$$

so the eigenvalues are $1, \pm i$, and we can find eigenvectors $(1,1,1)$, $(-1,i,1)$ and $(-1,-i,1)$, so

$$\mathbf{F}_n = A^{n-1}F_1 = X\Lambda^{n-1}X^{-1}F_1 = X\Lambda^{n-1}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 1^{n-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 i^{n-1}\begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} + c_3(-i)^{n-1}\begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}$$

for $\mathbf{c} = X^{-1}F_1$, but the latter is easier to calculate by substituting $n = 1,2,3$, rather than calculating X^{-1}.

So f_n is the first entry of the vector above, that is, $f_n = c_1 - c_2 i^{n-1} - c_3(-i)^{n-1}$. For $n = 1,2,3$ we get $c_1 - c_2 - c_3 = 1$, $c_1 - ic_2 + ic_3 = 2$ and $c_1 + c_2 + c_3 = 3$, which gives $c_1 = 2$, $c_2 = c_3 = \frac{1}{2}$, and

$$f_n = 2 - \frac{1}{2}i^{n-1} - \frac{1}{2}(-i)^{n-1}.$$

Then $f_{100} = 2 - \frac{1}{2}i^{99} - \frac{1}{2}(-i)^{99} = 2 - \frac{1}{2}i^3 - \frac{1}{2}(-i)^3 = 2 - \frac{1}{2}(-i) - \frac{1}{2}i = 2$, using that $i^4 = 1$.

An alternative solution is when we consider the vector space V of all sequences satisfying the given recursion. This is clearly 3-dimensional, since the sequences starting with $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ form a basis in it. We try to find another, nicer basis in V: sequences of the form $g_n = \lambda^n$ for some $\lambda \neq 0$. $g_n \in V \Leftrightarrow \lambda^n = \lambda^{n-1} - \lambda^{n-2} + \lambda^{n-3} \Leftrightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$. Since this polynomial has three different roots: $1, \pm i$ (note that these are the eigenvalues of the matrix of the recursion), we get three independent sequences of this form, thus a basis of V. Then every sequence satisfying the recursion, so also f_n, can be written az $c_1 1^n + c_2 i^n + c_3(-i)^n$, and we get the values of c_1, c_2, c_3 by substituting the initial values.

10. *Solve the differential equation $y''(t) = 3y'(t) - 2y(t)$ with initial values $y'(0) = 1$, $y(0) = 0$ using the matrix method.*

Solution: We rewrite the second order differential equation into a first order differential equation for vectors.

$$\text{for } Y(t) = \begin{bmatrix} y'(t) \\ y(t) \end{bmatrix} \text{ we have } Y'(t) = \begin{bmatrix} y''(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3y'(t) - 2y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y'(t) \\ y(t) \end{bmatrix} = AY(t).$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = 1,\ \lambda_2 = 2$$

Eigenvectors:

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \text{ gives } \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \text{ gives } \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

So for $X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, we have

$$Y(t) = e^{At}Y(0) = X \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} X^{-1}Y(0) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We can calculate the coefficients by substituting $t = 0$:

$$Y(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} c_1 & = & -1 \\ c_2 & = & 1 \end{matrix}$$

$$\text{So } Y(t) = -e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{2t} \\ -e^t + e^{2t} \end{bmatrix}, \text{ that is, } y(t) = -e^t + e^{2t}.$$

11. *Leftland and Rightland are neighbouring countries with open borders. Their population $l(t)$ and $r(t)$ satisfy (t measured in years)*

$$\begin{matrix} dl/dt & = & r - l \\ dr/dt & = & l - r \end{matrix}.$$

Comment on what phenomenon these describe. Suppose that all 10^8 people celebrate the Third Millennium at Leftland (with Rightland left empty) when they are subjected to these laws. What will be the situation after 10 years? Are these functions convergent at $t \to \infty$?

Solution: The equations show that the population of the bigger country is getting smaller, and that of the smaller country is getting bigger, and the rate of the change is proportionate with the difference in population. In fact, we may look at the functions $l(t) + r(t)$ and $l(t) - r(t)$ instead of $l(t)$ and $r(t)$. Then we see that the derivative of $l(t) + r(t)$ is zero, so the sum of the population is constant, while $(l(t) - r(t))' = -2(l(t) - r(t))$, so the difference is converging to 0 exponentially.

The differential equation for the population vector $P(t) = \begin{bmatrix} l(t) \\ r(t) \end{bmatrix}$ is

$$P'(t) = \begin{bmatrix} l'(t) \\ r'(t) \end{bmatrix} = \begin{bmatrix} r(t) - l(t) \\ l(t) - r(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} l(t) \\ r(t) \end{bmatrix} = AP(t)$$

$|A - \lambda I| = \lambda^2 + 2\lambda = \lambda(\lambda + 2)$, and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 0$, while $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -2$, so

$$P(t) = c_1 e^{0t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$P(0) = \begin{bmatrix} c_1 - c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 10^8 \\ 0 \end{bmatrix} \Rightarrow c_1 = 5 \cdot 10^7,\ c_2 = -5 \cdot 10^7, \text{ and}$$

$$P(t) = \begin{bmatrix} l(t) \\ r(t) \end{bmatrix} = 5 \cdot 10^7 \begin{bmatrix} 1 + e^{-2t} \\ 1 - e^{-2t} \end{bmatrix} \to \begin{bmatrix} 5 \cdot 10^7 \\ 5 \cdot 10^7 \end{bmatrix} \text{ when } t \to \infty.$$

$P(10) = 5 \cdot 10^7 \begin{bmatrix} 1 + e^{-20} \\ 1 - e^{-20} \end{bmatrix}$, and $e^{-20} \approx 2 \cdot 10^{-8}$, so the closest integer to $l(10)$ and $r(10)$ is already $5 \cdot 10^7$.

12. *Let* $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$. *Diagonalise* A *and determine* e^{At}.

Solution: The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0$, corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. With $X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, we have $X^{-1}AX = \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

$$e^{At} = Xe^{\Lambda t}X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{2t}-1 \\ 0 & 1 \end{bmatrix}.$$

But for this specific matrix it is also easy to calculate e^{At} as $\sum\limits_{n=0}^{\infty} \frac{1}{n!} t^n A^n$, since $A = 2B$ with $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $B^2 = B$, so $B^n = B$ for $n \geq 1$. We get that

$$\sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n = I + \sum_{n=1}^{\infty} \frac{1}{n!} t^n 2^n B = I + (e^{2t} - 1) \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{2t}-1 \\ 0 & 1 \end{bmatrix}.$$