1. *Determine the eigenvalues and an orthonormal set of eigenvectors of S and write it $S = Q\Lambda Q^T$.*

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution:

$$|S - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 4\lambda + 3) + (-(2-\lambda)) = (2-\lambda)(\lambda^2 - 4\lambda + 2),$$

so $\lambda_1 = 2$, $\lambda_{2,3} = 2 \pm \sqrt{2}$. Since there are 3 different eigenvalues, we get three orthogonal eigenvectors, which form a basis of $\mathbb{R}^3$. Eigenvectors:

$$S - 2I = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$S - (2+\sqrt{2})I = \begin{bmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \mapsto \begin{bmatrix} 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \\ -\sqrt{2} & -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \\ 0 & 1 & \sqrt{2} \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

$$S - (2-\sqrt{2})I = \begin{bmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{bmatrix} \mapsto \begin{bmatrix} 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \\ \sqrt{2} & -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \\ 0 & 1 & -\sqrt{2} \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

The normalized eigenvectors are

$$\mathbf{q}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \quad \Rightarrow \quad Q = \begin{bmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}.$$

$$\text{Then } S = \begin{bmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \\ 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix}.$$

2. *Find the/a Schur decomposition of*

$$\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}.$$

Solution: The eigenvalues of the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$ are the roots of $\begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$, that is, $\lambda_{1,2} = 3$. Eigenvector for $\lambda = 3$:

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{b}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Normalize this matrix and complete it to an orthonormal basis: $\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Then the first column of the conjugate of A by the orthogonal matrix $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ consisting of the vectors of $\mathcal{B}$ will be $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$, so it will be an upper triangular matrix:

$$Q^{-1}AQ = Q^TAQ = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} = T,$$

so $QTQ^{-1} = QTQ^T = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ is a Schur decomposition of A.

3. *True of false?*
(i) If A has n orthogonal eigenvectors then it has n orthonormal eigenvectors.
(ii) If A is real 2×2 and its determinant is negative then A has a positive and a negative pivot.
(iii) If A is real 2×2, its determinant is negative, and A has two orthonormal eigenvectors then A has a positive and a negative pivot.
(iv) If A is symmetric and $A^{100} = 0$ then $A = 0$.

Solution:
(i) True because eigenvectors cannot be zero, so they can be normalized.
(ii) False. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no pivots, but the determinant is -1.
(iii) False. The matrix A given for question (ii) has two orthonormal eigenvectors, since it is symmetric.
(iv) True. Since A is symmetric, A can be diagonalized: $A = X\Lambda X^{-1}$, so $\Lambda^{100} = X^{-1}A^{100}X = X^{-1}0X = 0$, thus $\lambda_i^{100} = 0$ for every diagonal element λ_i of $\Lambda \Rightarrow \lambda_i = 0$ for all $i \Rightarrow \Lambda = 0 \Rightarrow A = X\Lambda X^{-1} = 0$.

4. *Find an a so that the following matrix has a negative eigenvalue.*

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & a \\ 1 & a & 2 \end{bmatrix}.$$

How many negativ eigenvalues can A have? What are the signs of the pivots? What are the signs of the LPM's?

Solution: With elimination:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & a \\ 1 & a & 2 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & a \\ 0 & a & \frac{3}{2} \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & a \\ 0 & 0 & \frac{3}{2} - \frac{a^2}{2} \end{bmatrix}$$

The pivots are 2, 2 and $\frac{1}{2}(3 - a^2)$, so A has a negative pivot if and only if $|a| > \sqrt{3}$, say, for $a = 2$. In that case the signs of the eigenvalues are also $+, +, -$, and the signs of the LPM's are $+, +, -$, since the k'th LPM is the product of of the first k pivots.
We can check that for $a = 2$ we indeed have two positive and a negative eigenvalue:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 2 \\ 1 & 2 & 2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 4\lambda) - (2-\lambda) = (2-\lambda)(\lambda^2 - 4\lambda - 1),$$

so the eigenvalues are $2, 2 \pm \sqrt{5}$, two positive and one negative.

5. *Simon says: "If A is symmetric then $N(A)$ and $C(A)$ are orthogonal subspaces." Is he right? Why? He also says: "No positive definite matrix has a 0 on the main diagonal!" Is he right now? Why?*

Solution: Both statements are true. If A is symmetric then $C(A) = C(A^T)$, which is the orthogonal complement of $N(A)$. If A had a 0 on the diagonal, say, in the ith position then $\mathbf{e}_i^T A \mathbf{e}_i = a_{ii} = 0$ would contradict the assumption that A is positive definite.

6. HW *Find all orthogonal matrices that diagonalise*

$$A = \begin{bmatrix} 31 & -8 \\ -8 & 19 \end{bmatrix}$$

7. *Show that if a and b are chosen so that A and B are positive definite then C is also positive definite.*

$$A = \begin{bmatrix} 1 & 3 \\ 3 & a \end{bmatrix}; \qquad B = \begin{bmatrix} 4 & b \\ b & 6 \end{bmatrix}; \qquad C = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

Determine the decomposition $C = LDL^T$ *if* $a = 25$ *and* $b = 20$. *Use it to find* M *such that* $C = M^T M$ *(Cholesky factorisation).*

Solution: The LPM's of A are 1 and $a - 9$, and of B are 4 and $24 - b^2$, so A and B are positive definite if and only if $a > 9$ and $b^2 < 24$. But then the LPM's of C are also positive: $a = 9 > 0$ and $a^2 - b^2 > 81 - 24 = 57 > 0$, so C is also positive definite.
For $a = 25$ and $b = 20$:

$$C = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \mapsto \begin{bmatrix} 25 & 20 \\ 0 & 9 \end{bmatrix} = U \text{ with } L = \begin{bmatrix} 1 & 0 \\ 4/5 & 1 \end{bmatrix} \Rightarrow$$

$$C = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4/5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 20 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4/5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 4/5 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$C = \begin{bmatrix} 1 & 0 \\ 4/5 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4/5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 0 & 3 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 5 & 4 \\ 0 & 3 \end{bmatrix}$$

8. *Let*

$$S = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}.$$

What is the determinant of S, what are the eigenvalues and eigenvectors of S? Is S positive definite?

Solution: $S = X\Lambda X^T$ with $X = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ orthogonal matrix, where $\Lambda = \mathrm{diag}(4, 6)$. So $X^T = X^{-1}$, and $S = X\Lambda X^{-1}$, which implies that the diagonal elements of Λ are eigenvalues of S, and the columns of X, $\begin{bmatrix} \cos\vartheta \\ \sin\vartheta \end{bmatrix}$ and $\begin{bmatrix} -\sin\vartheta \\ \cos\vartheta \end{bmatrix}$ are eigenvectors corresponding to 4 and 6. S is a symmetric matrix because $S = X\Lambda X^T$ with Λ diagonal, and the eigenvalues of S are both positive, so S is positive definite. S is similar to Λ, so $|S| = |\Lambda| = 24$ (but it can also be obtained as the product of the eigenvalues, or as $|S| = |X| \cdot |\Lambda| \cdot |X^T| = 1 \cdot 24 \cdot 1 = 24$).

9. *Find the singular values and the SVD of* $A = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$. *What are the eigenvalues of A?*

Solution:

$$A^T A = \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix}, \quad |A^T A - \lambda I| = \lambda^2 - 25\lambda, \quad \begin{matrix} \lambda_1 & = & 25 & \sigma_1 & = & 5 \\ \lambda_2 & = & 0 & & & \end{matrix} \qquad \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

Eigenvectors of $A^T A$:

$$\lambda_1 = 25: \quad \begin{bmatrix} -20 & -10 \\ -10 & -5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 0: \quad \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$V = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, \quad AV = \begin{bmatrix} \sqrt{5} & 0 \\ -2\sqrt{5} & 0 \end{bmatrix}$$

The first $r(A)$ columns of U we get as $\mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$, that is, by normalizing the nonzero columns of AV. Then we complete these to an (arbitrary) orthogonal matrix U.

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{5}\begin{bmatrix} \sqrt{5} \\ -2\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} = U\Sigma V^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$|A - \lambda I| = \lambda^2 + 5\lambda \Rightarrow$ the eigenvalues of A are -5 and 0.
Actually, since A is symmetric, A can be diagonalized by an orthogonal matrix (ordering the eigenvalues so that their absolute values are decreasing), and then the decomposition $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ can be modified into an SVD by factoring Λ into a diagonal matrix with nonnegative elements and a diagonal matrix with ± 1 in the main diagonal.

$$A = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

10. *What are the singular values and the SVD of the matrix* $B = [\,1 \quad 2 \quad -3\,]$*?*
Solution:

$$B^T B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & 9 \end{bmatrix}$$

This matrix has rank 1, so $\dim N(B^T B) = 2$, hence the eigenvalues are $0, 0, \lambda$, where $\lambda = 0+0+\lambda = \operatorname{tr} B^T B = 14$, giving $\lambda = 14$. In decreasing order, $\lambda_1 = 14$, $\lambda_2 = \lambda_3 = 0$. The only singular value is $\sqrt{14}$, and $\Sigma = [\,\sqrt{14} \quad 0 \quad 0\,]$. The eigenvectors of $B^T B$:

$$\text{for } \lambda_1 = 14: \quad \begin{bmatrix} -13 & 2 & -3 \\ 2 & -10 & -6 \\ -3 & -6 & -5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -5 & -3 \\ -13 & 2 & -3 \\ -3 & -6 & -5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -5 & -3 \\ 0 & -63 & -42 \\ 0 & -21 & -14 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{special sol.:} \begin{bmatrix} -1/3 \\ -2/3 \\ 1 \end{bmatrix}. \text{ Let } \mathbf{b}_1 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

$$\text{for } \lambda_{2,3} = 0: \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & 9 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{special solutions: } \mathbf{b}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$\{\,\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\,\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors for $B^T B$, but the eigenvectors $\mathbf{b}_2, \mathbf{b}_3$ from the eigenspace for 0 are not orthogonal. We orthogonalize them:

$$\mathbf{c}_3 = \mathbf{b}_3 - \frac{\mathbf{b}_2 \cdot \mathbf{b}_3}{\|\mathbf{b}_2\|^2}\mathbf{b}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \frac{6}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}, \text{ or rather } \mathbf{c}_3 = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}.$$

So $\{\,\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_3\,\}$ is an orthogonal eigenbasis for $B^T B$, and

$$V = \frac{1}{\sqrt{70}}\begin{bmatrix} -\sqrt{5} & -2\sqrt{14} & 3 \\ -2\sqrt{5} & \sqrt{14} & 6 \\ 3\sqrt{5} & 0 & 5 \end{bmatrix}, \quad BV = \frac{1}{\sqrt{70}}[\,14\sqrt{5} \quad 0 \quad 0\,] = [\,\sqrt{14} \quad 0 \quad 0\,], \quad U_1 = [1] = U.$$

$$B = U\Sigma V^T = [1]\,[\,\sqrt{14} \quad 0 \quad 0\,]\,\frac{1}{\sqrt{70}}\begin{bmatrix} -\sqrt{5} & -2\sqrt{5} & 3\sqrt{5} \\ -2\sqrt{14} & \sqrt{14} & 0 \\ 3 & 6 & 5 \end{bmatrix}.$$