1. Let $M \in \text{Mod-}R$ be a right R-module, B a left ideal and J a right ideal of R, $a \in M$, and U, V submodules in M. Which of the following are necessarily submodules of M? (For the sets X, Y, the sum means $X + Y = \{\, x + y \,|\, x \in X, y \in Y \,\}$, the product $XY = \{\, \sum_i x_i y_i \,|\, x_i \in X,\ y_i \in Y\ \forall i \,\}$, while the annihilator Ann_M denotes the set of elements of M whose product with all the elements of the given subset of the ring is 0.)

a) aR b) aB c) aJ d) $U \cap V$ e) $U \cup V$

f) $U + V$ g) $\text{Ann}_M(B)$ h) $\text{Ann}_M(J)$ i) UB j) UJ.

2. What can be the additive group of a (unary) module over $\mathbb{Z}$, $\mathbb{Z}_3$ or $\mathbb{Z}_6$? Determine the number of 12-element modules over each ring up to isomorphism.

3. a) Let $1 \in S \leq R$. Prove that every R-module is also an S-module, but the converse is not true.

b) Suppose that $I \lhd R$. What is the connection between the modules over R and the modules over R/I?

4. Let $V = \mathbb{R}^n$ be an n-dimensional vector space over $\mathbb{R}$. Find a subring S of the ring of $n \times n$ matrices such that the only nontrivial S-submodule with respect to the usual vector-matrix multiplication is the following.

a) $U = \{(x_1, \ldots, x_n) \in V \,|\, x_1 + \ldots + x_n = 0\,\}$

b) $U = \{(x, \ldots, x) \in V \,|\, x \in \mathbb{R}\,\}$

5. Let A, B, C be submodules of M such that $A \geq C$. Prove that $A \cap (B + C) = (A \cap B) + C$.

6. Suppose that $N \leq M \in \text{Mod-}R$. Prove that M has a maximal submodule U such that $N \cap U = 0$, and that for such a module U, the intersection of $N \oplus U$ with any nonzero submodule of M is nonzero. Give an example among abelian groups to show that $N \oplus U$ is not necessarily the whole M.

7. Which of the following classes of modules have the property that every module can be written as a direct sum of cyclic, or of simple modules?

a) vector spaces

b) modules over a division ring

c) finite Abelian groups

d) Abelian groups

e) modules over $\mathbb{Z}_n$

f) modules over $K[x]$, where K is a field

HW1. Consider the following set M as a right module over the ring R.

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; a, b \in K \right\} \quad \text{és} \quad R = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \;\middle|\; x, y \in K \right\},$$

where K is a field. Prove that every submodule of M is also a K-subspace. Determine all the 1-dimensional submodules of M. How many such submodules exist if $K = \mathbb{Z}_5$?

HW2. Prove that the group algebra of a nontrivial, not necessarily finite group cannot be a division algebra. (Hint: Show that $\left\{ \sum_{g \in G} \lambda_g g \in KG \;\middle|\; \sum_{g \in G} \lambda_g = 0 \right\}$ is an ideal in KG.)