Repr. of Rings and Groups **Problem Set 2** *February 14, 2019*

1. Let $\varphi \in \mathrm{Hom}_R(M, N)$. Prove that

a) φ is surjective $\Leftrightarrow$ $(\varphi\alpha = \varphi\beta \;\Rightarrow\; \alpha = \beta)$ for all $\alpha, \beta \in \mathrm{Hom}_R(N, L)$;

a) φ is injective $\Leftrightarrow$ $(\alpha\varphi = \beta\varphi \;\Rightarrow\; \alpha = \beta)$ for all $\alpha, \beta \in \mathrm{Hom}_R(L, M)$;

2. Let $X, Y, Z \in$ Mod-R. Prove that $Y \cong X \oplus Z \Leftrightarrow$
$\exists\, X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y \underset{\delta}{\overset{\gamma}{\rightleftarrows}} Z$ such that $\alpha\gamma = 0$, $\beta\delta = 0$, $\alpha\beta = \mathrm{id}_X$, $\delta\gamma = \mathrm{id}_Z$, $\beta\alpha + \gamma\delta = \mathrm{id}_Y$.

3. Let $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \overset{\alpha}{\twoheadrightarrow} \mathbb{Z}_2 \overset{\beta}{\longleftarrow} \mathbb{Z} \oplus \mathbb{Z}$ such that $(x, y)\alpha = x + y$ and $(x, y)\beta = x$. Complete this into a commutative diagram with $\mathbb{Z} \oplus \mathbb{Z} \overset{\gamma}{\longrightarrow} \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in two ways, so that γ is is surjective in the first, but not surjective in the second.

4. Determine all the (finite) projective modules over $\mathbb{Z}_n$.

5. Prove the following two properties about injective modules, similarly to the proof of the corresponding properties of projective modules.

a) Every direct summand of an injective module is injective.

b) Any direct product of injective modules is injective.

6. Prove that $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module.

7*. Prove that every subgroup of a free abelian group is free. (Hint: Let $G = \bigoplus_{\alpha<\kappa} \langle g_\alpha \rangle$, where κ is a cardinality, and $G_\alpha = \bigoplus_{\beta<\alpha} \langle g_\beta \rangle$ for every ordinal number $\alpha < \kappa$. For a subgroup $H \leq G$, we define the subgroups $H_\alpha = H \cap G_\alpha$. Show that $H_{\alpha+1} \cong H_\alpha \oplus \mathbb{Z}$ or H_α for every α.)

HW1. Prove that for a right R-module M, the Abelian group $\mathrm{Hom}(R_R, M)$ is also a right R-module with the action of φr ($\varphi \in \mathrm{Hom}(R_R, M)$ and $r \in R$): $x(\varphi r) := (rx)\varphi$.

HW2. Determine the number of projective modules with at most 100 elements over the ring $\mathbb{Z}_{180}$ up to isomorphism. Give another ring R for which the given Abelian groups are also projective as R-modules but there are other R-projectives with at most 100 elements.