1. Suppose $e \in R$ is an idempotent element. Prove that $R_R = eR \oplus (1-e)R$.

2. Write R_R as a direct sum of indecomposable modules. What are the dimensions of the direct components?
a) $R = KC_4$, $K = \mathbb{Z}_2$
b) $R = KC_4$, $K = \mathbb{R}$
c) $R = KC_4$, $K = \mathbb{C}$
d) R is the ring of 3×3 upper triangular matrices over $\mathbb{R}$

3. Let $0 = M_0 < M_1 < \ldots < M_{k-1} < M_k = M$ be a composition series of the module M, and let U be a submodule of M. Prove that the factors of the series

$$0 = M_0 \cap U \leq M_1 \cap U \leq \ldots \leq M_{k-1} \cap U \leq M_k \cap U = U \text{ and}$$
$$0 = (M_0 + U)/U \leq (M_1 + U)/U \leq \ldots \leq (M_{k-1} + U)/U \leq (M_k + U)/U = M/U$$

are all simple or zero modules, and at each step the factor is zero in exactly one of the two series, and isomorphic to the corresponding factor M_i/M_{i-1} in the other.

4. Prove that
a) $\mathrm{Hom}(\bigoplus_{i\in I} M_i, N) \cong \prod_{i\in I} \mathrm{Hom}(M_i, N)$, and
b) $\mathrm{Hom}(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} \mathrm{Hom}(M, N_i)$.

5. Suppose that for a submodule $U \leq R_R$, the factor module R_R/U is semisimple. Prove that $U \geq J(R)$. Give an example when $R_R/J(R)$ is not semisimple.

A right or left ideal I of R is nilpotent if there is an integer $k > 0$ such that $I^k = 0$

6. Prove that the following three statements are equivalent for a right ideal J of a finite dimensional algebra A.
(i) J is nilpotent.
(ii) Every simple A-module is annihilated by J.
(iii) Every finite dimensional A-module is annihilated by an appropriate power of J.

7. Prove the following statements for the Jacobson radical of a finite dimensional algebra A.
a) $J(A)$ annihilates all (semi)simple modules.
b) $J(A)$ is the smallest right ideal such that $A_A/J(A)$ is semisimple (i.e. it is contained by all the other right ideals with this property).
c) $J(A)$ is the largest nilpotent right ideal (i.e. it contains all the other nilpotent right ideals).
d) $J(A)$ is the only right ideal with the property that $A_A/J(A)$ is semisimple and $J(A)$ is nilpotent.
e) $J(A)$ is a two-sided ideal.

8. Which of the rings in problem 2 are semisimple? What is the Jacobson radical in each case?

HW1. Suppose that M is a direct summand of R_R, i.e. $M = eR$ for some idempotent element e. Prove that M is decomposable if and only if there is an idempotent element $f \in eRe$ such that $0 \neq f \neq e$.

HW2. Prove that $\mathrm{Hom}(M, \bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I} \mathrm{Hom}(M, N_i)$ if M is finitely generated.